Collusion with Exhaustible Resources

Draft

Chad Hogan and Stephen P. Holland

April 27, 2002

1 Introduction

Collusion is thought to occur with some frequency in exhaustible resource industries. OPEC, probably the world's best known cartel, is widely believed to have maintained prices significantly above competitive prices for much of its thirty year history.[1] Since the members of OPEC are sovereign nation states, there is no international organization which could enforce any collusive agreement reached by the member states. Therefore any agreement reached by the member states must be enforced by credible punishments for defection from the agreement. Generally firms are thought to punish defections from a collusive agreement by increasing production to drive down the future profits of the defecting firm. However, exhaustibility of the resource base will hinder the cartel's ability to flood the market thus making it difficult to punish defection. This paper explores the effects of exhaustibility on the ability of firms to enforce collusive agreements.

Although the credible enforcement of collusive agreements is well understood in general, there is reason to believe that the standard analysis from infinitely repeated games does not simply extend to exhaustible resource industries. In addition to hindering punishment of defection, exhaustibility also implies that the strategic interaction may terminate at some date in the future.[2] When all firms but one have exhausted their deposits, the remaining firm will be a monopolist, and the equilibrium will be unique. If the stage game describing the firms' interaction had an unique equilibrium, the game might be solved by backwards induction over the state space to find a unique equilibrium. This reasoning suggests that not only may the standard analysis be inapplicable, but that collusion may not be supportable

[1] See Griffin [13], Adelman [3] and Griffin and Xiong [14].

[2] In the Rubinstein [22] bargaining game, the strategic interaction ends after a division of the surplus is agreed upon. Recall that the Rubinstein game has a unique subgame perfect equilibrium where the surplus is divided in the first period.

in industries with binding exhaustibility constraints.[3]

Although provocative, this reasoning turns out to be only partially correct, and this paper demonstrates that some collusion can be credibly supported in exhaustible resource industries. Although there is indeed a unique equilibrium for small states, this does not necessarily imply a unique equilibrium for larger states due to the non-concavity of the equilibrium value function for small states. The non-concavity leads to multiple Markovian equilibria for larger states, and these multiple equilibria can then be used as rewards and punishments to support a collusive agreement. Thus exhaustibility hinders, but does not prevent, credible enforcement of collusive agreements.

While Hotelling's [15] classic analysis characterizes competitive outcomes in resource markets, and monopolistic models of extraction are well understood (see, for example, Dasgupta and Heal [7]), the literature has been slow to develop understanding of the outcomes generated when intermediate forms of competition prevail. The major obstacle inhibiting characterization of imperfectly competitive outcomes has been the sheer complexity of identifying equilibria in the dynamic games that capture the competitive process. The typical response to this difficulty has been to generate outcomes using restrictive solution concepts. For example, the open-loop (or pre-commitment) solution identifies strategies which specify resource extraction as a function only of time. Such strategies are selected to be best responses to each other, hence such solutions constitute a restricted class of Nash equilibria. Alas, such a solution concept fails the credibility requirements of subgame perfection, despite its apparent tractability. On the other hand, the Markovian solution (specifying extraction behavior as a function only of remaining stocks) satisfies the requirements of subgame perfection, but restricts the range of strategic behavior available to players in the game. This paper provides the first attempt to characterize the full set of subgame perfect equilibria for a resource extraction model.

The extraction model studied in this paper is truly designed to represent an imperfectly competitive version of Hotelling's original model. Importantly, the resource is nonrenewable, and property rights to existing stocks of the resource are well defined amongst competitors. This latter feature distinguishes the model from the significant literature deal-

[3] This argument is presented in Mason and Polasky [17].

ing with common property appropriation of a resource in a dynamic setting (see for example Dutta [8], Sundaram [25], Amir [4], Reinganum and Stokey [21] and Eswaran and Lewis [9]).[4] The existing literature addressing the private property problem is less abundant. Salant [23],[24] initially raised the issue in relation to world oil markets. The "oil'igopoly" issue was pursued further by Loury [16]. Both authors used an open loop solution concept to describe outcomes in their games. Eswaran and Lewis [10] solve linear demand–quadratic cost and iso-elastic demand–zero cost versions of the model using the Markov perfect solution concept, and discuss numerical approximations to Markovian solutions in the game with various other functional forms.

Section 2 describes the assumptions which guarantee uniqueness of the stage game equilibrium. Section 3 introduces the infinite-horizon dynamic game with multiple finite deposits. The section relies heavily on the pioneering work of Abreu, Pearce and Stacchetti [1], [2] to show the upper hemi-continuity of the equilibrium value correspondence. The uniqueness of the equilibrium for small states is then demonstrated. To proceed further, Section 4 simplifies the model to two firms one with an exhaustible deposit and the other with an inexhaustible resource. This simplification implies that the state space is single dimensional and the game can be solved analytically by backwards induction. It is shown that the non-concavity of the equilibrium value function leads to multiple equilibria. Although the multiple continuation equilibria can be used to support some collusive agreements, it is shown that the monopoly payoff cannot be supported in the simple model. A simulation model is then presented in section 5 to illustrate the extent to which exhaustibility hinders collusion.

2 The Static Model

The static model of resource extraction describes the stage game that will be used in the dynamic model. Effectively, it constitutes a static Cournot oligopoly model, in which players are subject to capacity constraints.

The set of players is denoted $\mathbf{N} = \{1, 2 \ldots, N\}$, and each player $i \in \mathbf{N}$ has a payoff

[4] Note, however, that this literature almost uniformly restricts attention to Markovian solutions of the common property extraction game.

function

$$\pi_i(x_i, x_{-i}) = x_i P(x_i + \sum_{j \neq i} x_j) - c_i(x_i)$$

where P is the inverse demand function, c_i is the extraction cost function and s_i is the stock owned by player i. Each player, then, solves the problem

$$\max_{x_i \in [0, s_i]} \pi_i(x_i, x_{-i}) \tag{1}$$

where s_i is the quantity of the resource available to player i, and so represents the player's physical capacity constraint. The following assumptions affect the single-period payoff function:

Assumption 1 *The inverse demand function, $P : \mathbf{R}_+ \to \mathbf{R}_+$, and the cost functions, $c_i : \mathbf{R}_+ \to \mathbf{R}_+$, are continuous on the whole domain. P is non-increasing and c is convex.*

This assumption has two important implications. Firstly, each player's single-period payoff function is continuous, allowing application of the maximum theorem to each player's optimization problem. Secondly, there exists a price, $\overline{P}$, at which demand is zero. The importance of the latter requirement will be highlighted as required.

Assumption 2 *The payoff function, π_i, exhibits* ***strictly increasing differences*** *in $(x_i, -x_{-i})$ on $\mathbf{R}^N_+$, where the space is endowed with the standard ordering.*

A function f is said to exhibit increasing differences in (s,t) if, for all $(s'', t'') \geq (s', t')$

$$f(s'', t') - f(s', t') \leq f(s'', t'') - f(s', t'').$$

If the weak inequalities are replaced with strict inequalities, then f has strictly increasing differences. Hence, if π_i exhibits strictly increasing differences in $(x_i, -x_{-i})$, then player i's marginal profit decreases in the output of all other players. If the profit function were twice continuously differentiable, this would be equivalent to[5]

$$\frac{\partial^2 \pi_i(x_i, x_{-i})}{\partial x_i x_j} < 0 \quad \forall\, j \neq i.$$

[5] This follows from Topkis' Characterization Theorem. See Topkis [26], p 42.

Assumption 2, then, is the non-differentiable version of the Novshek condition, commonly used to guarantee existence of pure strategy equilibrium in Cournot oligopoly games.[6] An equivalent interpretation of this assumption is as a sufficient condition to ensure that the Cournot oligopoly game is truly a game in which players' outputs are strategic substitutes (i.e. best response functions are decreasing in others' outputs).

It is well known that lattice theoretic approaches provide powerful tools for analysis of games exhibiting strategic complementarities. In contrast, games of strategic substitutability have not generally been amenable to lattice theoretic analysis. This paper, however, will employ lattice theoretic techniques to analyze the quantity competition game in the presence of strategic substitutability, and this approach will provide particularly strong results by guaranteeing uniqueness of equilibrium. Notice that, for each player $i \in \mathbf{N}$, we can restate the payoff function as

$$\hat{\pi}_i(y_i, x_{-i}) = (y_i - \sum_{j \neq i} x_j)P(y_i) - c_i(y_i - \sum_{j \neq i} x_j)$$

simply by making the substitution

$$\hat{\pi}_i(y_i, x_{-i}) = \pi_i(y_i - \sum_{j \neq i} x_i, x_{-i}).$$

Using this alternative form of the payoff function, we can restate each player's maximization problem (1) as

$$\max_{y_i} \ \hat{\pi}_i(y_i, x_{-i}) \quad s.t. \ y_i \in [\sum_{j \neq i} x_j, \sum_{j \neq i} + s_i] \tag{2}$$

The solutions to (1) and (2), x_i^* and y_i^* respectively, must satisfy

$$y_i^* = \sum_{j \neq i} x_j + x_i^*.$$

In parallel to Assumption 2, the following assumption places structure on the function $\hat{\pi}_i$.

Assumption 3 *The single period payoff function $\hat{\pi}_i(y_i, x_{-i})$ exhibits strictly increasing differences in (y_i, x_{-i}), when the domain, $\mathbf{R}_+^N$, is endowed with the standard ordering.*

[6] Novshek [19] assumes that cost functions are twice continuously differentiable and convex in own output, and that the inverse demand function is twice continuously differentiable with $x_i \partial^2 P / \partial x_i^2 + \partial P / \partial x_i < 0$. This latter condition, known as the Novshek condition, is sufficient to ensure that the cross-partial of the profit function is negative.

This assumption implies that increments in total output will be more beneficial to player i when competitors' output is high. Notice that the experiment of increasing x_{-i} while maintaining a given level of y_i requires that player i reduce her own output to offset the increase in x_{-i}. Hence, the requirement that the marginal value of y_i increase with x_{-i} simply requires that marginal units of output be more valuable when own output is low and others' output is high. Alternatively stated, the impact that own output has on marginal profit must be greater than the impact that others' output has on marginal profit. The requirement that a player's own actions have greater impact upon own marginal values than other players' actions seems quite reasonable, not only in this scenario but in a wide range of economic environments. More explicitly, in the case of Cournot competition in the market for a homogeneous product the condition can be seen to be extremely weak. Consider a market in which cost curves and the demand curve are twice continuously differentiable. Recalling the twice continuously differentiable characterization of the increasing differences property, assumption 3 is equivalent to

$$\frac{\partial^2 \hat{\pi}_i}{\partial y_i x_j} = \frac{\partial^2 \pi_i}{\partial x_i x_j} - \frac{\partial^2 \pi_i}{\partial x_i^2} > 0.$$

It is easily verified that this is satisfied as long as

$$P'(X) - c_i''(x_i) < 0$$

and this is clearly guaranteed by Assumption 1. In the event that the payoff function is not known to be twice continuously differentiable, this condition does provide slightly more structure than the simple implication that the difference between price and marginal cost decreases. In fact, this assumption seems to require that the payoff function be continuously differentiable. To see this, observe firstly that Assumptions 2 and 3 imply that the payoff function must be strictly concave in x_i. Therefore, the function defining marginal profit must be decreasing. Hence, there can be no upward jumps in the marginal profit function. Assumption 3 additionally implies that the marginal profit function may have no downward jumps. Therefore, the marginal profit function must be continuous and decreasing.

3 The Infinite Horizon Dynamic Model

The dynamic model studied here is a Cournot extraction model played out in discrete time. In each period, $t = 1, 2 \ldots$, players simultaneously select quantities of output, $x_{i,t} > 0 \ \ \forall i$, to sell on the market. If $X_t = \sum_{i=1}^{N} x_{i,t}$, then the market price at time t will be given by

$$P_t = P(X_t).$$

Period t profits for player i are given by

$$\pi_{i,t}(x_t) = x_{i,t} P(X_t) - c_i(x_{i,t}).$$

where x_t is the vector of players' period t extraction quantities. Alternatively, profits are captured by

$$\hat{\pi}_{i,t}(y_{i,t}, x_{-i,t}) = (y_{i,t} - \sum_{j \neq i} x_{j,t}) P(y_{i,t}) - c_i(y_{i,t} - \sum_{j \neq i} x_{j,t})$$

where $y_{i,t}$ is the total output chosen by player i in response to the output vector $x_{-i,t}$ from other players. The average discounted value of profits for player i over the infinite time horizon is

$$\Pi_i(\{x_t\}_{t=1}^{\infty}) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \pi_i(x_{i,t}, x_{-i,t})$$

where $\delta \in (0, 1)$ is the common discount factor.

Each player's choice of quantity in any period is made subject to constraint. In period t, the total resource stock available to player i is given by $s_{i,t}$, and player's extraction choices are constrained to ensure that

$$x_{i,t} \in [0, s_{i,t}]$$

or

$$y_{i,t} \in [\sum_{j \neq i} x_{j,t}, \sum_{j \neq i} x_{j,t} + s_{i,t}].$$

Furthermore, stocks in the model vary deterministically according to the obvious law of motion

$$s_{i,t} = s_{i,t-1} - x_{i,t},$$

given initial stock levels of $s_{i,0}$. The *state* of the system in period t will be described by the vector of stocks players carry from period t into period $t+1$:

$$s_t = (s_{1,t}, s_{2,t} \dots, s_{N,t}).$$

and the *history* of the game in period-t is given by the sequence of states observed from period 0 to period t:

$$h_t = (s_0, s_1, s_2 \dots s_t).$$

Notice that all extraction information can be deduced from this series of state variables. A period-t history is *feasible* if the state weakly decreases over time, and if the state remains non-negative. Denote the set of all feasible histories (given the initial vector of stocks, s_0) by

$$H(s_0) = \{\ h_t\ |0 \leq s_\tau \leq s_{\tau-1}\ \ \forall \tau = 1, 2 \dots t \text{ and } \forall\ t = 1, 2 \dots \infty\}.$$

Histories will be observable to all players when extraction choices are made in each period. Hence, a player's strategy will specify the quantity she will extract in period-t as a function of the observed history:

$$\sigma_i : H(s_0) \rightarrow \mathbf{R}_+,$$

and the set of strategies available to player i is given by

$$\Sigma_i = \{\sigma_i : \ \ \forall\ h_t \in H(s_0),\ \sigma_i(h_t) \in [0, s_{i,t}]\}.$$

A strategy profile is given by

$$\sigma = (\sigma_1, \sigma_2 \dots \sigma_N)$$

where each $\sigma_i \in \Sigma_i$.

The *equilibrium value correspondence* will map each state, s into the set of average discounted value N-tuples that players may receive in subgame perfect equilibria of the game with initial state s:

$$V : \mathbf{R}_+^\mathbf{N} \rightarrow \mathbf{R}^\mathbf{N}.$$

We can also identify the set of payoffs available to player i in subgame perfect Nash equilibrium:

$$V_i(s) = \text{proj}_i V(s).$$

In order to characterize the full set of subgame perfect Nash equilibria in this game, we first describe the equilibrium value correspondence. In fact, the equilibrium value correspondence will be shown to be single valued for some states, which will imply existence of a unique subgame perfect equilibrium for those states.

3.1 Upper Hemi-continuity of the Equilibrium Value Correspondence

The discussion of the equilibrium value correspondence makes extensive use of the following generalizations of tools introduced by Abreu, Pearce and Stacchetti [1], [2].

Firstly, let $W : \mathbf{R}_+ \to \mathbf{R^2}$ be a correspondence which we can think of as a candidate equilibrium value correspondence. A pair (q, κ) is said to be *admissible* with respect to W at state s if it satisfies the following properties:

1. $q \in \mathbf{R}^2_+ \quad \text{such that} \quad q_1 \leq s$
2. $\kappa : \mathbf{R}^2_+ \to \mathbf{R^2} \quad \text{such that} \quad \kappa(q) \in W(s - q_1) \;\; \forall\, q \text{ s.t. } q_1 \leq s$
3. $(1-\beta)\pi_1(q) + \beta\kappa_1(q) \geq (1-\beta)\pi_1(q_1', q_2) + \beta\kappa_1(q_1', q_2) \;\; \forall\, q_1' \leq s_1$
4. $(1-\beta)\pi_2(q) + \beta\kappa_2(q) \geq (1-\beta)\pi_2(q_1, q_2') + \beta\kappa_2(q_1, q_2') \;\; \forall\, q_2' \in \mathbf{R}_+.$

A new correspondence may be derived from the first:

$$B(W) : \; \mathbf{R}_+ \;\to\; \mathbf{R^2}$$

which is given by

$$B(W)(s) = \left\{ (w_1, w_2) \;\middle|\; \begin{array}{l} w_i = (1-\beta)\pi_i(q) + \beta\kappa_i(q) \text{ for } i = 1, 2; \text{ and} \\ (q, \kappa) \text{ is admissible with respect to } W \text{ at state } s \end{array} \right\}.$$

If $W(s) \subseteq B(W)(s) \;\; \forall\, s$, then we say that $W(s)$ is *self generating.* The following results follow directly from Abreu, Pearce and Stacchetti [2].

Self Generation Theorem: If $W(s)$ is self generating, then $B(W)(s) \subseteq V(s) \;\; \forall\, s$.

Factorization Theorem: $V(s) = B(V)(s) \;\; \forall\, s$.

Using these results, we can prove Lemma 1.

Lemma 1 *The equilibrium value correspondence, V, is upper hemi-continuous.*

Proof. Define

$$\text{graph}(W) := \{(x,y) \in \mathbf{R}_+ \times \mathbf{R}^2 \ : \ y \in W(x)\}.$$

for any correspondence $W : \mathbf{R}_+ \to \mathbf{R}^2$. Berge [6], chapter 6, establishes that the correspondence W is upper hemi-continuous if graph(W) is closed and W is bounded. We will use this characterization of upper hemi-continuity throughout this proof.

Firstly, suppose that graph(W) is closed and W is bounded (i.e. W is u.h.c). We can show that $B(W)$ must be upper hemi-continuous.

To establish that $B(W)$ is bounded, consider any pair $(s, w) \in \text{graph}(B(W))$. There must exist some pair (q, κ) which is admissible with respect to W such that

$$w_i = (1-\beta)\pi_i(q) + \beta\kappa_i(q) \quad i = 1, 2$$

where $\kappa(q) \in W(s - q_1)$. By assumption π_i is bounded; and as W is u.h.c., $W(s - q)$ is bounded. Hence $B(W)$ is bounded.

Next, we establish that graph($B(W)$) is closed. Consider some convergent sequence

$$\{(s^t, w^t)\} \to (s^*, w^*)$$

such that $(s^t, w^t) \in \text{graph}(B(W))\ \forall\, t$. Then, for each t there exists some pair, (q^t, κ^t), that is admissible with respect to W at s^t. As the graph of W is compact in any compact region containing all s^t, without loss of generality we can choose the sequence so that $\{(q^t, \kappa^t(q^t))\}$ is convergent. Let $\{(q^t, \kappa^t(q^t))\} \to (q^*, K)$.

Define the function κ^* to satisfy the following

$$\kappa^*(q) := \begin{cases} K & \text{if } q = q^* \\ \underline{w}^i(s^* - q_1) & \text{if } q_i \neq q_i^* \textit{ and } q_{-i} = q_{-i}^* \\ \in W(s^* - q_1) & \textit{otherwise} \end{cases}$$

where

$$\underline{w}^i(s') := \arg\min\{w_i : w \in W(s')\}$$

which is well defined as $W(s')$ is compact.

As $(s^t - q_1^t, \kappa^t(q^t)) \in \text{graph}(W)\ \forall\ t$ and as graph(W) is closed and bounded, we obtain that $\lim(s^t - q_1^t, \kappa^t(x^t)) = (s^* - q_1^*, K) \in W(s^* - q_1^*)$. Hence, κ^* is a well defined continuation value function for s^*. We want to show that the pair (q^*, κ^*) is admissible with respect to W at s^*. This means we will need to show that q is a Nash equilibrium extraction pair when continuation values are defined by κ^*.

First, consider firm 1. Notice that, for any q_1, the sequence $\kappa^t(q_1, q_2^t)$ is contained in a compact region. Without loss of generality, this sequence converges to $\tilde{K}$ which is in $W(s^* - q_1)$ due to the upper hemi-continuity of W. As we have defined $\kappa^*(q_1, q_2^*) = \underline{w}^i(s^* - q_1)$ for all $q_i \neq q_i^*$ it must be true that $\kappa_1^*(q_1, q_2^*) \leq \tilde{K}_1$.

A similar argument applies for firm 2. For any value of q_2 there is a convergent subsequence $\{\kappa^t(q_1^*, q_2)\}$ which converges to $\hat{K}$. As W is u.h.c., $\hat{K} \in W(s^* - q_1^*)$. By definition of κ^*, we have $\kappa_2^*(q_1^*, q_2) \leq \hat{K}_2$.

Hence, for any $\epsilon > 0$ there exists a t sufficiently large so that $\kappa_i^*(q_i, q_{-i}^*) < \kappa_i^t(q_i, q_{-i}^t) + \epsilon/4$. For any $\epsilon > 0$ and any q_1, then, there exists some T such that for all $t > T$

$$\begin{aligned}(1-\beta)\pi_i(q_i, q_{-i}^*) + \beta\kappa_i^*(q_i, q_{-i}^*) &< (1-\beta)\pi_i(q_i, q_{-i}^t) + \beta\kappa_i^t(q_i, q_{-i}^t) + \epsilon/2 \\ &\leq (1-\beta)\pi_i(q^t) + \beta\kappa_i^t(q^t) + \epsilon/2 \\ &< (1-\beta)\pi_i(x^*) + \beta\kappa_i^*(x^*) + \epsilon\end{aligned}$$

where the first inequality follows from the continuity of π_i and the argument presented immediately preceeding the set of inequalities; the second inequality follows from the admissibility of (q^t, κ^t) at s^t; and the final inequality follows from the continuity of π_i and the fact that $\{(q^t, \kappa^t(q^t))\} \to (q^*, \kappa^*(q^*))$. As $\epsilon \to 0$ we obtain

$$(1-\beta)\pi_i(q_i, q_{-i}^*) + \beta\kappa_i^*(q_i, q_{-i}^*) \leq (1-\beta)\pi_i(q^*) + \beta\kappa_i^*(q^*).$$

which establishes that (q^*, κ^*) is admissible with respect to W at s^*. Furthermore,

$$(1-\beta)\pi(x^*) + \beta K = \lim[(1-\beta)\pi(x^t) + \beta\kappa^t(x^t)] = \lim w^t = w^*.$$

Therefore, $(s^*, w^*) \in \text{graph}B(W)$, which shows that $B(W)$ has the closed graph property. $B(W)$, then, is upper hemi-continuous.

Now consider the equilibrium value correspondence, V. It is clear that V is bounded. Any $v \in V(s)$ for any s is derived as the average discounted value of an infinite stream of profits. As the per period profit level is bounded above by the maximum single period revenue, it must be that the average discounted profit level is bounded above. Similarly, the ability of a firm never to extract resources from its stock provides a straightforward individual rationality constraint which bounds equilibrium payoffs from below.

Define the correspondence $\mathrm{cl}(V)$ so that $\mathrm{graph}(\mathrm{cl}(V))$ is the closure of $\mathrm{graph}(V)$. As $\mathrm{graph}(\mathrm{cl}(V))$ is closed and $\mathrm{cl}(V)$ is bounded (by virtue of the boundedness of V), we conclude that $\mathrm{cl}(V)$ is upper hemi-continuous. Employing the result above, it is also apparent that $B(\mathrm{cl}(V))$ is upper hemi-continuous.

Furthermore, $\mathrm{graph}(V) \subseteq \mathrm{graph}(B(\mathrm{cl}(V)))$. Consider any pair, $(s, v) \in \mathrm{graph}(V)$. Then there exists some pair, (q, κ), which is admissible with respect to V at s. This requires that $\kappa(q') \in V(s - q_1') \ \forall\ 0 \leq q_1' \leq s$. This implies that $\kappa(q') \in \mathrm{cl}(V)(s - q_1')\ \forall 0 \leq q_1' \leq s$ as well. It must be true, then, that (q, κ) is admissible with respect to $\mathrm{cl}(V)$ at s, so that $(s, v) \in \mathrm{graph}(B(\mathrm{cl}(V)))$.

As $\mathrm{graph}(V) \subseteq \mathrm{graph}(B(\mathrm{cl}(V)))$ and $\mathrm{graph}(B(\mathrm{cl}(V)))$ is closed, we can conclude that $\mathrm{graph}(\mathrm{cl}(V)) \subseteq \mathrm{graph}(B(\mathrm{cl}(V)))$, i.e., $\mathrm{cl}(V)$ is self generating. By the Self-Generation Theorem, $\mathrm{graph}(\mathrm{cl}(V)) \subseteq \mathrm{graph}(V)$. Thus, $\mathrm{graph}(V)$ is closed, and V is bounded. ■

3.2 Uniqueness of Subgame Equilibria

After all the deposits are exhausted, the equilibrium of all remaining subgames is trivially unique. Thus the equilibrium value correspondence is single valued at zero for the state with no stock. Upper hemi-continuity of V implies that values of V for states near zero must also be near zero. This limits the potential for rewarding cooperation and punishing defection for small stocks. Since rewards and punishments are limited, the following proposition shows that there is an unique equilibrium for small states.

Proposition 1 *Suppose that assumptions 1–3 are satisfied. Then there exists some real number* $s^* > 0$ *such that*

$$s^* = \inf_i \{s_i : s_i = sup\{s : \frac{\partial \pi_i(s,0)}{\partial x_i} > \delta \frac{\partial \pi_i(0,0)}{\partial x_i}\}\}.$$

Furthermore, if the state of the system, s_t, is such that

$$\sum_{i=1}^{N} s_{i,t} \leq s^*$$

then the equilibrium value correspondence, V, maps s_t into a unique equilibrium payoff vector, and $V_i(s) = (1-\delta)\pi_i(s)$ for all i.

Proof

The first part of the lemma is established by the fact that $\frac{\partial \pi_i(x_i, x_{-i})}{\partial x_i}$ is continuous in x_i for all i and for all $x \in \mathbf{R}_+^N$. Notice, it is important here that there exists some price at which all demand is choked off (Assumption 1), or the derivative $\frac{\partial \pi_i(0,0)}{\partial x_i}$ would not be defined.[7]

In order to establish the second part of the lemma, consider any sequence of feasible extraction quantities from time t onwards for producers $j \neq i$, $\{x_{-i,\tau}\}_{\tau=t}^{\infty}$. Player i receives an average discounted payoff of

$$\pi_i(\{x_\tau\}_{\tau=t}^{\infty}) = (1-\delta)\pi_i(s_{i,t}, x_{-i,t})$$

if she extracts her whole stock, $s_{i,t}$, in period t. If producer i withholds a marginal unit of stock from the market in period t to sell it in period $t' > t$, then the variation in player i's average discounted payoff will be

$$\Delta_i = -(1-\delta)\frac{\partial \pi_i(s_{i,t}, x_{-i,t})}{\partial x_i} + \delta^{t'-t}(1-\delta)\frac{\partial \pi_i(0, x_{-i,t'})}{\partial x_i}.$$

Given that $\sum_{j \neq i} x_{j,t} + s_{i,t} \leq \sum_{j=1}^{N} s_{j,t} \leq s^*$, we can conclude that

$$\frac{\partial \pi_i(s_{i,t}, x_{-i,t})}{\partial x_{i,t}} > \frac{\partial \pi_i(s^*, 0)}{\partial x_i} \geq \delta \frac{\partial \pi_i(0,0)}{\partial x_i} \geq \delta^{t'-t}\frac{\partial \pi_i(0,0)}{\partial x_i} \geq \delta^{t'-t}\frac{\partial \pi_i(0, x_{-i,t'})}{\partial x_i}$$

for all $t' > t$. The first inequality follows from Assumption 2 and 3; the second from the definition of s^*; the third from the fact that $\delta < 1$; and the fourth from Assumption 2. Thus,

$$\Delta = -(1-\delta)\frac{\partial \pi_i(s_{i,t}, x_{-i,t})}{\partial x_{i,t}} + \delta^{t'-t}(1-\delta)\frac{\partial \pi_i(0, x_{-i,t'})}{\partial x_{i,t'}} < 0.$$

This establishes that it is a dominant strategy for each player in this subgame to extract all remaining resources in period-t. Thus, if the state s_t lies in the region defined by $\sum_{j=1}^{N} s_{i,t} \leq s^*$, then there is a unique subgame perfect Nash equilibrium and therefore a unique vector of subgame perfect equilibrium payoffs. ■

[7] The role of the choke price should be appreciated here. If prices were able to rise arbitrarily high, then players would always have incentive to retain stock. The game would not reach the steady state and it would be impossible to establish uniqueness of equilibrium in the range identified in the lemma.

4 The Simple Extraction Model

Proposition 1 demonstrates that the infinite horizon dynamic model has a unique equilibrium for states with small stocks. Since the game could be solved by backwards induction over the state space, the uniqueness of the equilibrium for small states demonstrated in Proposition 1 suggests that the resulting equilibrium would be unique for the entire game. This would imply that tacit collusion could not be enforced with credible threats of punishment. In fact, this logic does not apply due to the non-concavity of the single-valued portion of the equilibrium value correspondence. This non-concavity implies multiple Markovian equilibria and thus the infinite horizon dynamic model has multiple subgame perfect equilibria.

To demonstrate the multiplicity of equilibria, consider a simplification of the infinite horizon dynamic model. The simple extraction model has only two firms. Firm 1 extracts the resource from a finite deposit of size $\bar{S}$ while firm 2 uses a renewable technology to produce a perfect substitute.[8] Both firms have the common discount factor β and seek to maximize intertemporal profits. Let q_i represent production by firm i in the stage game, and $\pi_i(q_1, q_2)$ be firm i's stage game profit. Assume the π_i's are continuously differentiable, bounded, and satisfy the Novshek conditions so that a stage game equilibrium exists and is unique. At the end of each period, the firms observe each others' actions and adjust their production accordingly.

The unused stock of firm 1 completely describes the state of the dynamic system. In particular, note that all subgames which begin with state $s \in [0, \bar{S}]$ look identical to each firm, i.e., all subgames have the same potential for future rewards and punishments regardless of the history.[9] Label the subgame beginning with state s *subgame* s. Define the equilibrium value correspondence $V : \mathbf{R}_+ \rightarrow \mathbf{R}^2$ mapping the state into the set of average discounted value payoffs which can be attained in a subgame perfect equilibrium of the subgame which begins with that state. The following lemmas characterize the equilibrium

[8] The simple extraction model approximates a model where one firm has a deposit which is much larger than the other firm's deposit.

[9] Note that we am not imposing the Markovian restriction that strategies be independent of the history. Rather, we are merely observing that any strategy profile which is an equilibrium for a subgame beginning with state s would also be an equilibrium for every subgame beginning with state s regardless of the history which led to the subgame.

value correspondence.

Lemma 2 *If the stock is positive, firm 1's initial extraction is non-zero in any SPNE.*

Proof. Suppose there exists an equilibrium strategy profile, σ, with no extraction from firm 1 in the first period. Since on the equilibrium path the state is the same in the first two periods, an equilibrium can be constructed which repeats σ in the second period on the equilibrium path, i.e., firm 1 does not extract anything in the second period either. Similarly, an equilibrium can be constructed where firm 1 never extracts anything and thus receives no profits. Since in this equilibrium firm 2 must be producing such that the price is always zero, firm 2 also has zero profits. However, firm 2 could increase profits by decreasing production in any period. ■

Consider the interval A on which V is single-valued. Note that subgame 0 has a unique subgame perfect equilibrium where firm 1's payoff is zero and firm 2 receives monopoly profits in every subsequent period. Since this subgame has a unique SPNE, $V(0) = (0, \pi^m)$ is single-valued, and the interval A is non-empty. Recall that a (stationary) Markov perfect equilibrium restricts strategies to be functions only of the state. Let B be the interval on which there is a unique MPE. Note that when the state is zero, the subgame perfect equilibrium identified above is also a unique MPE. Thus, the interval B is also non-empty. The following lemma can now be stated:

Lemma 3 *If a MPE exists and $A \neq \emptyset$, then $A = B$.*

Proof. Since any Markov equilibrium is also subgame perfect, $A \subseteq B$. Suppose $A \subset B$. Since V is uhc by Lemma 1 and single-valued on A, there exist $s \in A$, $s' \in B$, ϵ, δ, ϵ_1, and q_2^* such that: $s' \notin A$, $|s - s'| < \min\{\epsilon, \epsilon_1\}$, $\frac{\partial \pi_2}{\partial q_2}(0, q_2^*) = -\delta$, $\frac{\partial \pi_1}{\partial q_1}(\epsilon_1, q_2^*) = \delta$, and $\beta \| V(s'') - V(s''') \| < \delta$ for every s'' and s''' in $[s, s')$. Consider possible continuations for the equilibria of the subgame s'. Suppose there exists some equilibrium of subgame s' with continuation not in A. This equilibrium specifies initial extraction $q_1 < \min\{\epsilon, \epsilon_1\}$ and production q_2. By Lemma 2, $q_1 > 0$. If firm 1 were to increase current extraction by an incremental unit, it would receive a marginal increase in current profits but a loss in the future as the continuation switches to the punishment phase. By assumption, the loss from switching to the punishment phase is bounded by δ. Since this marginal deviation for firm 1 must not be profitable

$$\delta \geq \frac{\partial \pi_1}{\partial q_1}(q_1, q_2) > \frac{\partial \pi_1}{\partial q_1}(\epsilon_1, q_2)$$

which implies that $q_2 > q_2^*$. Note that this implies that firm 2's marginal profit is negative. But now consider firm 2's incentive to deviate. If firm 2 decreases production by a marginal unit, it avoids the current loss on the marginal unit and the equilibrium switches to the punishment phase. This benefit from decreasing production is

$$-\frac{\partial \pi_2}{\partial q_2}(q_1, q_2) > -\frac{\partial \pi_2}{\partial q_2}(0, q_2^*) = \delta$$

Since the loss in the punishment phase is limited by δ, this deviation for firm 2 is profitable. Thus no equilibrium of subgame s' can have its continuation state not in A.

Since every equilibrium of subgame s' must have its continuation in A, both firms' best responses must solve the Bellman equations where the continuation correspondence is single valued. The uniqueness of the Markov equilibrium implies that these equations have a unique solution and thus that subgame s' has a unique subgame perfect equilibrium. ■

This lemma implies that we can restrict our attention to Markov perfect equilibria since uniqueness of the Markov equilibrium implies uniqueness of the subgame perfect equilibrium. The next lemma helps describe the Markovian equilibria:

Lemma 4 *In any subgame s, firm 2's Markovian best response in the first period is its stage-game best response.*

Proof. Suppose firm 2's strategy does not produce the stage-game best response in the first period. Firm 2 could increase current profits by producing the stage-game best response. Since the continuation pay-off is unaffected by this deviation, the deviation is profitable. ■

Lemma 4 greatly simplifies the computation of Markov perfect equilibria since it completely describes firm 2's Markovian best response. Firm 1's optimization problem is more difficult. Let v be firm 1's value function. Since v must satisfy the Bellman optimality condition, the Markov perfect equilibrium is found by the simultaneous solution of the Bellman equation: eq:Bell

$$v(s) = \max_q \{\pi_1(q, q_2) + \beta v(s - q)\} \tag{3}$$

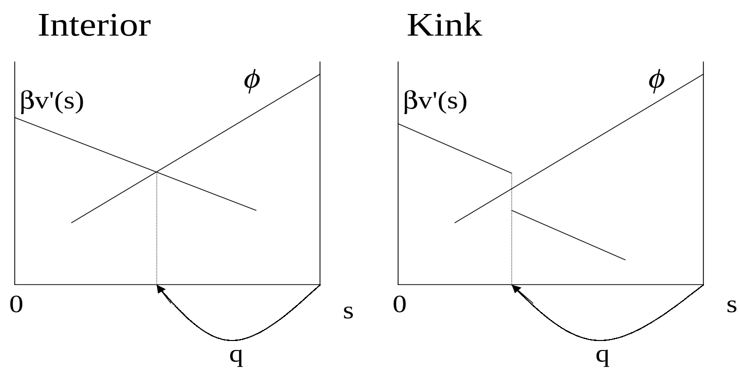

Figure 1: Interior and kink solutions to the Bellman equation with state s. Note that the policy function q is represented by the arrow illustrating the state transition.

and firm 2's optimality condition $q_2 = BR_2(q)$ where BR_2 is firm 2's stage-game best response function. Let q be the policy function which solves the Bellman equation, i.e., q maps the state s into firm 1's equilibrium best response in state s.[10]

The Bellman equation has two types of solutions: *interior* and *kink* solutions. The solution is interior if the first order condition of the Bellman equation: eq:intFO

$$\frac{\partial \pi_1}{\partial q_1}(q, BR_2(q)) = \beta v'(s-q) \tag{4}$$

holds.[11] The solution is a kink solution if there exists some q such that[12] eq:kink

$$\beta v'(s-q)^- > \frac{\partial \pi_1}{\partial q_1}(q, BR_2(q)) > \beta v'(s-q)^+ \tag{5}$$

This condition requires that the value function be non-differentiable (i.e., that it be "kinked") at the continuation state. Interior and kink solutions are illustrated in Figure 1. In both panels, firm 1's stock increases along the horizontal axis, and the current stock level is indicated by s. Concavity of the value function implies that $\beta v'$ is downward sloping. Define

[10]Existence of a Markovian equilibrium follows from firm 1's optimization problem.

[11]Note that for the solution to be interior, the value function must be differentiable at the continuation. The theorem of the maximum ensures that v is piece-wise continuously differentiable.

[12]The superscripts $^-$ and $^+$ indicate limits from the left and right, i.e., $v'(s)^- \equiv \lim_{x \uparrow s} v'(x)$.

the function $\phi(q) \equiv \frac{\partial \pi_1}{\partial q_1}(q, BR_2(q))$. Like the marginal profit function, ϕ is decreasing,[13] but it is flatter than the marginal profit function since ϕ implies an equilibrium response by firm 2. Since larger current extraction implies a smaller stock in the future, ϕ as drawn in Figure 1 slopes down to the left. If $\beta v'$ is continuous, then the solution will be interior as shown in the left panel. However, if $\beta v'$ is not continuous, the solution can be a kink solution as illustrated in the right panel. Note that here firm 1 does not want to increase current extraction since the profit from extracting the incremental unit today is less than the discounted value of extracting the unit tomorrow. Likewise, firm 1 does not want to decrease current extraction since the value of extracting the marginal unit today is greater than the value of extracting the unit tomorrow.

For a subgame s, let the equilibrium be an *interior* or *kink equilibrium* depending on whether the solution to the Bellman equation at state s is interior or a kink. Note that in a kink equilibrium, an incremental increase in the stock would be extracted immediately since equation 5 would still hold. Thus at a kink equilibrium, the policy function extracts each additional unit of stock immediately, i.e., $q'(s) = 1$.

If the equilibrium is interior on an open interval, the slope of the policy function can be found by differentiating equation 4. Differentiation yields:

$$\phi'(q)q'(s) = \beta v''(s - q(s))(1 - q'(s))$$

which implies eq:qslope

$$q'(s) = \frac{\beta v''(s-q)}{\phi'(q) + \beta v''(s-q)} \tag{6}$$

for an interior equilibrium. Since this is strictly less than one for $\phi' < 0$, the policy function at an interior equilibrium implies that the incremental unit of stock is extracted partly in the first period and partly in the continuation.

The slope of the value function can also be found at an interior equilibrium by differentiating equation 3 when it is optimized. Differentiation implies[14]

$$v'(s) = (\frac{\partial \pi_1}{\partial q_1} + \frac{\partial \pi_1}{\partial q_2} BR_2')q' + \beta v'(s-q)(1-q')$$

[13] Work on this.

[14] Note that this derivation illustrates that the envelope theorem does not hold for the equilibrium value function since $v'(s) \neq \beta v'(s-q)$.

$$= \frac{\partial \pi_1}{\partial q_2} BR_2' q' + \beta v'(s-q)$$

eq:vslope

$$= \frac{\partial \pi_1}{\partial q_1} + \frac{\partial \pi_1}{\partial q_2} BR_2' q' \tag{7}$$

This equation gives a convenient formula for computing the slope of the value function if it is differentiable. Note that this formula also holds for a corner equilibrium since at a corner equilibrium any increment in stock would affect extraction only in the first period and not in the continuation. Thus the slope of the value function at a corner equilibrium is

$$v'(s) = \frac{\partial \pi_1}{\partial q_1} q' + \frac{\partial \pi_1}{\partial q_2} BR_2' q'$$

Since $q' = 1$ in a corner equilibrium, the formula in equation 7 still holds. lem:kinks

Lemma 5 *If the equilibrium policy function is continuous at s, the equilibrium value function has a concave (convex) kink at s if the left derivative of the policy function is greater (less) than the right derivative at s.*

Proof: eq:vslope

$$v'(s)^- - v'(s)^+ = \frac{\partial \pi_1}{\partial q_2} BR_2' [q'(s)^- - q'(s)^+] \quad \blacksquare \tag{8}$$

Define s_i as the largest state for which all the stock is exhausted in at most i periods in a Markov equilibrium. Note that for $s \in [0, s_1]$, the value function, $v(s) = \pi_1(s, BR_2(s))$, is concave. How large is s_1? Firm 1 will extract the entire stock today if the marginal profit of extracting the entire stock is greater than the discounted marginal profit of extracting a small amount tomorrow. Thus, s_1 is given by: eq:s1def

$$\phi(s_1) = \beta v'(0) \tag{9}$$

Similarly s_2 is defined by eq:s2def

$$\phi(s_2 - s_1) = \beta v'(s_1)^+ \tag{10}$$

Define $\hat{s}_1$ by eq:hats1d

$$\phi(\hat{s}_1 - s_1) = \beta v'(s_1)^- \tag{11}$$

For $s \in [s_1, \hat{s}_1]$ the subgame s has an interior equilibrium since βv is continuously differentiable and concave at the continuation. By equation 5, subgame s has a corner equilibrium for $s \in (\hat{s}_1, s_2)$.

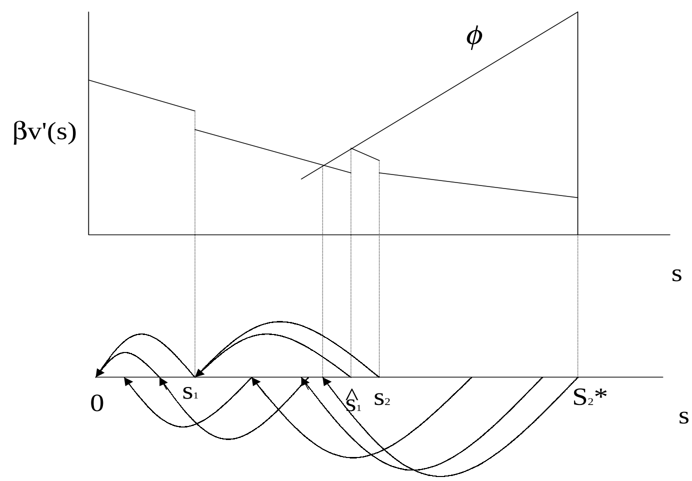

Figure 2: Multiple solutions to the Bellman first order conditions for subgame s_2^*. The policy function $q(s)$ is illustrated by the arrows showing the state transition. Note that the continuation $\hat{s}_1$ is not an equilibrium continuation of s_2^*.

Lemma 6 *The value function has a concave kink at* s_1 *and a convex kink at* $\hat{s}_1$ *iff* $\phi' < 0$.[15]

Proof. The condition implies $q' < 1$ for $s \in (s_1, \hat{s}_1)$. Thus by Lemma 5, the value function has a concave kink at s_1. This implies that $\hat{s}_1 < s_2$. Since $q' = 1$ for $s \in (\hat{s}_1, s_2)$, Lemma 5 implies that there is a convex kink at $\hat{s}_1$. ■

If the value function is concave, the maximization in the Bellman equation 3 yields a unique policy function. However, the convexity at $\hat{s}_1$ implies that the Bellman first order conditions in equation 4 may have multiple solutions. The smallest state, s_2^*, where equation 4 does not have a unique solution is defined by $\phi(s_2^* - \hat{s}_1) = \beta v'(\hat{s}_1)^+$. These two solutions, $s_2^* - \hat{s}_1$ and $q(s_2^*)^-$, are illustrated in Figure 2. Note however that $\hat{s}_1$ is not an equilibrium continuation of s_2^* since extraction of a marginal unit has a higher value in the present than in the continuation. Thus subgame s_2^* has a unique equilibrium. However, for stocks slightly larger than s_2^*, the marginal deviation is no longer profitable but larger deviations may be profitable. For any stock s slightly greater than s_2^*, there are two solutions

[15] I'm being a little sloppy here. To be more precise, I need this expression to be bounded away from zero at $q(s_1)^+$ and $q(\hat{s}_1)^-$. This condition holds trivially in the linear example since $-3/2 < 0$. I also suspect that this condition can be established from the conditions guaranteeing uniqueness of the stage-game Cournot equilibrium, but I have not explored this yet.

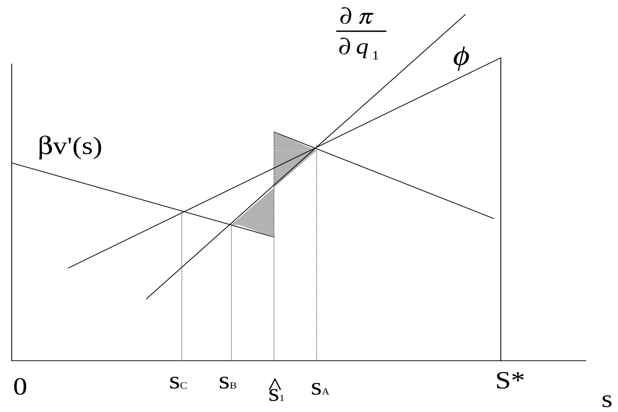

Figure 3: Multiple equilibria for subgame s^* with continuations at s_A and s_C. Note that the shaded triangles are equal and thus the optimal deviation to s_B is not profitable.

to the Bellman first order conditions in equation 4. Let the continuations of these solutions be s_A and s_C defined by the equations $\phi(s - s_A) = \beta v'(s_A)$ and $\phi(s - s_C) = \beta v'(s_C)$ where $s_C < \hat{s}_1 \leq s_A$ (see Figure 3). Is s_A an equilibrium continuation from s? Let s_B be the optimal deviation from s_A defined by $\frac{\partial \pi_1}{\partial q_1}(s - s_B, BR_2(s - s_A)) = \beta v'(s_B)$. Note that as s increases, the gain from this optimal deviation

$$\int_{s_B}^{s_A} \frac{\partial \pi_1}{\partial q_1}(s - \alpha, BR_2(s - s_A)) - \beta v'(\alpha) d\alpha$$

is decreasing since extracting each incremental unit where the continuation is greater than $\hat{s}_1$ decreases the profit from the deviation but extracting the incremental units which yield a continuation smaller than $\hat{s}_1$ increase the profit from the deviation. For the stock s^* illustrated in Figure 3, the optimal deviation yields no additional profit since the shaded triangles are equal. Thus both s_A and s_C are equilibrium continuations. Define $s^* > s_2^*$ as the smallest stock from which the optimal deviation is zero. Since the continuations s_A and s_C are both equilibrium continuations, we have the following lemma:

Lemma 7 *The subgame s^* has two Markov perfect equilibria in pure strategies.*

Proof. As shown above, there is a unique equilibrium on $[0, s^*)$. At s^*, extracting $s^* - s_A$ or $s^* - s_C$ solves the first order condition in equation 4. Since there are no profitable deviations by the definition of s_A, they are both equilibrium extraction quantities with firm 2 producing $BR_2(s^* - s_A)$ and $BR_2(s^* - s_C)$. ■

Since there are multiple Markov equilibria for subgame s^*, the equilibrium value correspondence is not necessarily single valued. In fact Figure 3 illustrates that $V(s^*)$ has multiple values. Consider firm 1's payoff in the equilibrium with continuation s_A. This payoff is given by

$$\int_0^{s_A} \beta v'(\alpha)d\alpha + \int_{s_A}^{s^*} \frac{\partial \pi_1}{\partial q_1}(s^* - \alpha, BR_2(s^* - s_A))d\alpha = \int_0^{s_B} \beta v'(\alpha)d\alpha + \int_{s_B}^{s^*} \frac{\partial \pi_1}{\partial q_1}(s^* - \alpha, BR_2(s^* - s_A))d\alpha$$

where the equality follows from the definition of s^*. Note that this payoff is clearly less than the payoff in the equilibrium with continuation s_C

$$\int_0^{s_C} \beta v'(\alpha)d\alpha + \int_{s_C}^{s^*} \frac{\partial \pi_1}{\partial q_1}(s^* - \alpha, BR_2(s^* - s_C))d\alpha$$

Thus firm 1's worst equilibrium payoff decreases discontinuously at s^*. Because $V(s^*)$ is not single valued, firm 2's stage game best response is no longer its unique best response since it may be able to credibly influence the continuation payoffs by choosing a different production level. Thus for subgames s with $s > s^*$, there are multiple subgame perfect equilibria. This demonstrates the proposition:

Proposition 2 *The simple extraction model has multiple subgame perfect equilibria and the equilibrium value correspondence is not single valued for stocks weakly greater than s^*.*

Proof. As described above, $V(s^*)$ is not single valued. For larger stocks, the possibilities for rewards and punishments contingent on observed actions implies that there are multiple subgame perfect equilibria and V is not single-valued. ■

Since there are multiple subgame perfect equilibria, it follows that some degree of collusion can be supported as a subgame perfect equilibrium. The folk theorem implies that in infinitely repeated games any individually rational payoffs, including payoffs yielding the joint monopoly profit, can be supported by a subgame perfect equilibrium. The uniqueness of the subgame perfect equilibrium for small states will limit the ability of firms to support collusive payoffs. The following proposition demonstrates how the uniqueness result hinders collusion.

Proposition 3 *The monopoly payoff cannot be supported by a subgame perfect equilibrium in the simple extraction model.*

Proof. Let σ be a strategy profile which yields the monopoly payoff. If σ transitions into the interval $(0, s_2^*)$ on the transition path, the unique stage-game equilibrium output is greater than the monopoly output. The Bellman optimality principle implies that σ must yield the monopoly profit level at each state along the transition path. Thus, if σ transitions into the interval $(0, s_2^*)$ on the transition path, σ is not subgame perfect. Suppose σ does not transition into $(0, s_2^*)$ on the transition path. If σ does not exhaust the stock, then σ is not subgame perfect since firm 1 could increase profits by extracting more stock. Suppose σ extracts the entire stock for some state $s > s_2^*$ on the transition path. For this subgame s, σ must require that firm 2 initially produce $BR_2(s)$ since the continuation is unique. However, firm 1 could increase profits by reducing extraction since $\beta v'(0) = \phi(s_1) > \phi(s)$. Therefore σ is not subgame perfect. ■

Proposition 3 shows that the monopoly payoff cannot be supported by a collusive agreement with credible punishment strategies. This result is somewhat surprising based on the well-known intuition of the folk theorem. Applications of the folk theorem usually show that the monopoly payoff can be supported at reasonable discount factors in infinitely repeated games. In fact, there are generally many subgame perfect equilibria which yield the monopoly payoff. This implies that the firms have many options for dividing the monopoly payoff between them with the options generally being bounded by the individual rationality constraints of the firms. However, in the simple extraction model the monopoly payoff cannot be supported by any subgame perfect equilibria. Note that this result holds for all discount factors. This suggests that it will be even more difficult to support all individually rational payoffs since all individually rational payoffs can only be approached in infinitely repeated games as the discount factor approaches unity.

5 Simulation

Propositions 2 and 3 suggest that limited collusion can be supported in the simple extraction model. However, the question remains as to whether these limitations would severely hinder attempts to support collusive agreements. To illustrate the limitations exhaustibility places on the collusive possibilities, consider the following numerical example of the simple extraction model where demand is $Q = 10 - P$ and there are no costs of extraction or production. Clearly the monopoly price is \$5 and the average discounted monopoly profit is \$25.

Ideally, the equilibrium value correspondence would be computed and compared to the joint profit maximizing outcome. However, the boundary of the equilibrium value correspondence might be quite difficult to define. Therefore, following Phelan and Stacchetti [20], only a convex bound of the equilibrium value correspondence is computed. For example, the simplest convex bound is defined by the maximum and minimum continuation values for each firm. This defines a box around the equilibrium value set for each state.

To simplify calculation of the convex bound, the state space and action space are approximated by a grid with size 0.01.[16] For each state there are a finite number of actions for each firm. Given the convex bound of the equilibrium value correspondence for all continuation states, an "equilibrium" can be computed by testing each pair of actions for the two firms.[17] An action pair is called an equilibrium if neither firm wants to deviate unilaterally given that it earns continuation payoffs which are contained in the convex bound of the equilibrium value correspondence. In the simple case of the box around the equilibrium value correspondence, all equilibria can be found by testing whether a firm wants to unilaterally deviate from the proposed action given that it earns the worst possible continuation payoff (worst punishment) for deviating and the best possible continuation payoff (best reward) for not deviating. For each firm, all possible equilibrium payoffs are compared to calculate the bounds of the equilibrium value correspondence for the current state. Given these bounds, the convex bound for the next largest state can be computed. In the simple case of the box, the maximum (*minimum*) payoff in each state becomes the upper (*lower*) bound of the box for that state.

The convex bound of the equilibrium value set can be described in a variety of ways which may be more or less useful depending on the question under consideration.

[16] Thus the relevant grid for each state has 1000×1000 grid points.

[17] The "equilibria" computed in the simulation are not necessarily subgame perfect equilibria since they promise continuation payoffs which may not be equilibrium payoffs. In particular, although the equilibrium promises a continuation payoff to firm 1 which is achieved in some equilibrium and a payoff to firm 2 which is achieved in some equilibrium, the two payoffs may not be attained in the same equilibrium.

Figure 4: Bounds on the equilibrium value correspondence for firm 1.

Although the box is a particularly simple way of describing the convex bound, it is less useful for determining the highest joint profit which can be attained by the firms since it promises an equilibrium continuation payoff in which the joint profit is larger than can be supported.[18] Therefore, in the simulation, the simple convex bound of the equilibrium value correspondence has been constrained such that the combined payoffs are less than the highest joint payoff which is actually attained in equilibrium. Computation of the convex bound is as for the box, except now each equilibrium must be tested to see that it can be supported by payoffs which promise continuation payoffs which can be jointly attained.[19]

The results of the simulation are presented in the figures 4-9. The simulation results differ from the results of section 4 since the action space is discrete and the equilibrium value correspondence is only approximated by the convex bound. The discrete action space means that the stage game equilibrium is not necessarily unique. This implies that the bounds on the equilibrium value correspondence may have multiple values for smaller states than s^*. Figure 4 shows the bounds for firm 1 when $\beta = 0.9$. When firm 1 has no stock, the equilibrium value is unique at zero. As firm 1's stock increases, the value it can attain in equilibrium generally increases. Calculations based on the results of section 4 imply that for the continuous action model $s_1 = 0.333$, $\hat{s}_1 = 0.8667$ and $s_2^* = 1.52$. Since the equilibrium value correspondence is single valued for states smaller than $s^* > s_2^*$, the multiple values of the bounds for states smaller than 1.52 arise from the multiple equilibria caused by the discrete action space. Note that figure 4 shows downward jumps in the minimum bound for firm 1. Section 4 showed that the minimum value of the equilibrium value correspondence for firm 1 jumps down at s^*. However, the jumps in figure 4 occur for larger states than s^*.

Figure 5 shows the bounds for firm 2 when $\beta = 0.9$. When firm 1 has no stock, firm 2 can attain the monopoly profit in every period for an average discounted payoff of \$25. As firm 1's stock increases, the maximum profit attainable by firm 2 decreases.

[18]The defection payoffs are supported since we can assume that the punishing firm would want to drive the deviating firm to its worst outcome.

[19]The simulation tests a finite number of continuation payoffs where the continuation payoffs divide the maximum joint payoff.

Figure 5: Bounds on the equilibrium value correspondence for firm 2.

Figure 6: Bounds on the equilibrium value correspondence for firm 2 for larger stocks.

Note in particular that for positive stocks firm 2 cannot attain the average discounted monopoly profit in equilibrium although it could produce the monopoly output forever. Thus competition from the severely constrained firm 1 reduces the potential profit to firm 2. Figure 6 shows the bounds for firm 2 for larger stocks. The figure demonstrates that competition from firm 1 drives down the highest equilibrium profit for firm 2 as the stock of firm 1 increases.

Figure 7 shows the bound for the combined value of the two firms when β is 0.9 and 0.95. Without exhaustibility constraints, the firms could collude to attain the average discounted monopoly profit in every period. Figure 7 illustrates that the firms cannot achieve the monopoly profit in all periods. As firm 1's stock increases, competition first decreases the maximum combined profit attainable by the two firms. As the stock becomes more abundant, the possibilities for collusion increase and the maximal combined profit increases. The upper bound on combined profit attains the average discounted monopoly profit for large stocks although Proposition 3 indicates that the monopoly profit is not attained by the equilibrium value correspondence.

Figure 7 also shows the affect of the discount factor on collusion. When firms are more patient, increased competition does not drive down the combined profits as far for a given stock. In addition, the firms are able to collude at lower stock levels and thus the combined bound for the two firms approaches the monopoly profit level for smaller stocks. This illustrates that firms are able to collude more when they are more patient.

The discount factor also affects the payoffs which can be attained by the individual firms. Figures 8 and 9 illustrate this effect. Without exhaustibility constraints, any individually rational payoffs could be supported as the discount factor approaches unity.

Figure 7: Maximum combined value for the two firms for $\beta = 0.9$ and $\beta = 0.95$.

Figure 8: Bounds on V_1 for $\beta = 0.9$ and $\beta = 0.95$.

Figure 9: Bounds on V_2 for $\beta = 0.9$ and $\beta = 0.95$.

Figures 8 and 9 suggest an analogous result with exhaustibility constraints. The worst individually rational payoff for firm 1 is zero and the best individually rational payoff for firm 2 is \$25. Figure 8 illustrates that as the firms become more patient, the best and worst payoffs decrease for firm 1 driving it toward its worst individually rational payoff of zero. On the other hand, patience helps firm 2 since its best and worst payoffs increase as the firms become more patient. Thus as the discount factor increases, the best payoff for firm 2 approaches its highest individually rational payoff of \$25.

6 Conclusion

Credible enforcement of collusive agreements is well-understood in the theory of infinitely repeated games. However, because of exhaustibility, the insights about competition and collusion from the theory of infinitely repeated games may not be applicable. This paper demonstrates that exhaustibility will hinder a cartel's enforcement of collusive agreements but will not prevent the cartel from supporting prices above the competitive level.

Collusion is hindered by exhaustibility since the strategic interaction ends when all firms but one exhaust their deposits. At this point, it becomes impossible for the cartel to reward or punish good or bad behavior. Prior to exhaustion, it also may be impossible to support collusion since the firms recognize that the continuation equilibrium is unique and thus rewards and punishments are impossible. However, as the state space becomes larger, the unique continuation equilibrium no longer assures a unique subgame equilibrium even under strict assumptions about uniqueness of equilibrium in the stage game. This follows because the equilibrium value function is not concave over the range where the equilibrium is unique. Concavity of the equilibrium value function would assure uniqueness of equilibrium in the dynamic game. However, in the dynamic game, concavity is not assured as it would be in single-agent dynamic optimization. Indeed, the equilibrium value function is not concave, and this non-concavity leads to multiple solutions to the firm's optimization problem and thus multiple (Markovian) equilibria. Multiple continuation equilibria then can be used to support collusion for larger states.

Exhaustibility hinders a cartel's ability to collude and may even prevent the cartel from supporting the monopoly outcome. However, the simulation results suggest that exhaustibility may not seriously hinder the cartel. In particular, the simulated bounds on the joint profit maximization approach the monopoly profit level for stocks which are quite small relative to demand. The results also illustrate the range of outcomes which can be supported for various discount factors. They suggest that patience may improve equilibrium outcomes for firms endowed with large stocks while worsening the outcomes for small firms.

References

[1] Abreu, Dilip, David Pearce and Ennio Stacchetti (1986), "Optimal Cartel Equilibria with Imperfect Monitoring", *Journal of Economic Theory* 39(1), 251–69.

[2] Abreu, Dilip, David Pearce and Ennio Stacchetti (1990), "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring", *Econometrica* 58(5), 1041–63.

[3] Adelman, M. (1986), "Scarcity and World Oil Prices," *Review of Economics and Statistics* 68, 387–397.

[4] Amir, Rabah (1989), "A Lattice-Theoretic Approach to a Class of Dynamic Games", *Computers and Mathematics With Applications*, 17(8–9), 1345–49.

[5] Amir, Rabah (1996), "Cournot Oligopoly and the Theory of Supermodular Games", *Games and Economic Behavior* 15(2), 132–48.

[6] Berge, Claude (1997), *Topological Spaces*, Dover Publications, Mineola, New York.

[7] Dasgupta, Partha and Geoffrey Heal (1979), *Economic Theory and Natural Resources*, Cambridge University Press, Cambridge, England.

[8] Dutta, Prajit and Rangarajan Sundaram (1993), "The Tragedy of the Commons?", *Economic Theory* 3(3), 413–426.

[9] Eswaran, Mukesh and Tracy Lewis (1984), "Appropriability and the Extraction of a Common Property Resource", *Economica* 51(24), 393–400.

[10] Eswaran, Mukesh and Tracy Lewis (1985), "Exhaustible Resources and Alternative Equilibrium Concepts", *Canadian Journal of Economics* 18(3), 459–73.

[11] Fudenberg, Drew and David Levine (1988), "Open-Loop and Closed-Loop Equilibria in Dynamic Games with Many Players", *Journal of Economic Theory* 44(1), 1–18.

[12] Gaudet, Gerard and Stephen Salant (1991), "Uniqueness of Cournot Equilibrium; New Results from Old Methods", *Review of Economic Studies*, 58(2), 399–404.

[13] Griffin, James (1985), "OPEC behavior," *American Economic Review*, 75, 954–963.

[14] Griffin, J. and W. Xiong (1997), "The Incentive to Cheat: An Empirical Analysis of OPEC," *Journal of Law and Economics* 40, 289–316.

[15] Hotelling, Harold (1931), "The Economics of Exhaustible Resources", *Journal of Political Economy*, 39(2), 137-75.

[16] Loury, Glenn (1986), "A Theory of 'Oil'igopoly': Cournot Equilibrium in Exhaustible Resource Markets with Fixed Supplies", *International Economic Review* 27(2), 285–301.

[17] Mason, Charles and Stephen Polasky (1998), "Non-Renewable Resource Cartels: Who's in the Club?" *mimeo*.

[18] Milgrom, Paul and John Roberts (1990), "Rationalizability, Learning and equilibrium in Games with Strategic Complementarities", *Econometrica* 58(6), 1255–1278.

[19] Novshek, William (1985), "On The Existence of Cournot Equilibrium", *Review of Economic Studies* 52(1), 85–98.

[20] Phelan, Chris and Ennio Stacchetti, "Sequential Equilibria in a Ramsey Tax Model," *Econometrica*, forthcoming.

[21] Reinganum, Jennifer and Nancy Stokey (1985), "Oligopoly Extraction of a Common Property Natural Resource: The Importance of the Period of Commitment in Dynamic Games", *International Economic Review* 26(1), 161–73.

[22] Rubinstein, Ariel (1982), "Perfect Equilibrium in a Bargaining Model", *Econometrica* 50, 97–110.

[23] Salant, Stephen (1976), "Exhaustible Resources and Industrial Structure: A Nash Cournot Approach to the World Oil Market", *Journal of Political Economy* 84(5), 1079–93.

[24] Salant, Stephen (1982), "Imperfect Competition in the International Energy Market: A Computerized Nash Cournot Model", *Operations Research* 30(2), 252–280.

[25] Sundaram, Rangarajan (1989), "Perfect Equilibrium in Non-randomized Strategies in a Class of Symmetric Dynamic Games", *Journal of Economic Theory* 47(1), 153–77.

[26] Topkis, Donald (1998), *Supermodularity and Complementarity*, Princeton University Press, Princeton, New Jersey.

[27] Vives, Xavier (1990), "Nash Equilibrium with Strategic Complementarities", *Journal of Mathematical Economics* 19(3), 305–21.