Proof of Correctness for Prim’s Algorithm

This handout refers to Prim’s algorithm as given in the Hein Discrete Structures book.

**Theorem 1** If $S$ is the spanning tree selected by Prim’s algorithm for input graph $G = (V, E)$, then $S$ is a minimum-weight spanning tree for $G$.

**Proof:** The proof is by contradiction, so assume that $S$ is not minimum weight. Let $ES = (e_1, e_2, \cdots, e_{n-1})$ be the sequence of edges chosen (in this order) by Prim’s algorithm, and let $U$ be a minimum-weight spanning tree that contains edges from the longest possible prefix of sequence $ES$.

Let $e_i = \{x, y\}$ be the first edge added to $S$ by Prim’s algorithm that is not in $U$, and let $W$ be the set of vertices immediately before $\{x, y\}$ is selected. Notice that it follows that $U$ contains edges $e_1, e_2, \cdots, e_{i-1}$ but not edge $e_i$.

There must be a path $x \sim y$ in $U$, so let $\{a, b\}$ be the first edge on this path with one endpoint ($a$) inside $W$, and the other endpoint ($b$) outside $W$, as in the following picture:

```
  a
  |
  b
  |
 W
  |
  x
  |
  y
```

Define the set of edges $T = U + \{\{x, y\}\} - \{\{a, b\}\}$, and notice that $T$ is a spanning tree for graph $G$. Consider the three possible cases for the weights of edges $\{x, y\}$ and $\{a, b\}$:

**Case 1**, $w(\{a, b\}) > w(\{x, y\})$: In this case, in creating $T$ we have added an edge that has smaller weight than the one we removed, and so $w(T) < w(U)$. However, this is impossible, since $U$ is a minimum-weight spanning tree.

**Case 2**, $w(\{a, b\}) = w(\{x, y\})$: In this case $w(T) = w(U)$, so $T$ is also a minimum spanning tree. Furthermore, since Prim’s algorithm hasn’t selected edge $\{a, b\}$ yet, that edge cannot be one of $e_1, e_2, \cdots, e_{i-1}$. This implies that $T$ contains edges $e_1, e_2, \cdots, e_i$, which is a longer prefix of $ES$ than $U$ contains. This contradicts the definition of tree $U$.

**Case 3**, $w(\{a, b\}) < w(\{x, y\})$: In this case, since the weight of edge $\{a, b\}$ is smaller, Prim’s algorithm will select $\{a, b\}$ at this step. This contradicts the definition of edge $\{x, y\}$.

Since all possible cases lead to contradictions, our original assumption (that $S$ is not minimum-weight) must be invalid. This proves the theorem.  □