Local periods and binary partial words: An algorithm

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Abstract

The study of the combinatorial properties of strings of symbols from a finite alphabet (also referred to as words) is profoundly connected to numerous fields such as biology, computer science, mathematics, and physics. Research in combinatorics on words goes back roughly a century. There is a renewed interest in combinatorics on words as a result of emerging new application areas such as molecular biology. *Partial words* were recently introduced in this context. The motivation behind the notion of a partial word is the comparison of genes (or proteins). Alignment of two genes (or two proteins) can be viewed as a construction of partial words that are said to be compatible. While a word can be described by a total function, a partial word can be described by a partial function. More precisely, a partial word of length n over a finite alphabet A is a partial function from $\{1, \ldots, n\}$ into A. Elements of $\{1, \ldots, n\}$ without an image are called holes. A word is just a partial word without holes. The notion of *period* of a word is central in combinatorics on words. In the case of partial words, there are two notions: one is that of *period*, the other is that of *local period*. This paper extends to partial words with one hole the well known result of Guibas and Odlyzko which states that for every word u, there exists a word v of same length as u over the alphabet $\{0, 1\}$ such that the set of all periods of u coincides with the set of all periods of v. Our result states that for every partial word u with one hole, there exists a partial word v of same length as u with at most one hole over the alphabet $\{0, 1\}$ such that the set of all periods of u coincides with the set of all periods of v and the set of all local periods of u coincides with the set of all local periods of v. To prove our result, we use the technique of Halava, Harju and Ilie which they used

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to characterize constructively the set of periods of a given word. As a consequence of our constructive proof, we obtain a linear time algorithm which, given a partial word with one hole, computes a partial word with at most one hole over the alphabet $\{0, 1\}$ with the same length and the same sets of periods and local periods. A World Wide Web server interface at http://www.uncg.edu/mat/AlgBin/ has been established for automated use of the program.

Keywords: Words, partial words, periods, and local periods.

1 Introduction

The study of the combinatorial properties of strings of symbols from a finite alphabet, also referred to as *words*, is profoundly connected to numerous fields such as biology, computer science, mathematics, and physics.

Research in combinatorics on words goes back roughly a century [12, 13]. There is a renewed interest in combinatorics on words as a result of emerging new application areas such as molecular biology. *Partial words* were recently introduced by Berstel and Boasson [1] in this context. The motivation behind the notion of a partial word is the comparison of two genes (or two proteins). Alignment of two such strings can be viewed as a construction of two partial words that are said to be compatible in a sense that will be described in Section 2.2. While a word can be described by a total function, a partial word can be described by a partial function. More precisely, a partial word of length n over a finite alphabet A is a partial function from $\{1, \ldots, n\}$ into A. Elements of $\{1, \ldots, n\}$ without an image are called holes. A word is just a partial word without holes.

The notion of *period* of a word is central in combinatorics on words. There are many fundamental results on periods of words. Among them is the well known periodicity result of Fine and Wilf [8] which intuitively determines how far two periodic events have to match in order to guarantee a common period. This result was extended to partial words with one hole by Berstel and Boasson [1]. In our recent work [2, 4], we extend this result further, with the exclusion of a few pathological cases, to partial words with an arbitrary number of holes. Another fundamental result on periods of words is the well known and unexpected result of Guibas and Odlyzko [9] which states that the set of all periods of a word is independent of the alphabet size (alphabets with one symbol are excluded here). In other words, for every word u, there exists a word v over the alphabet $\{0, 1\}$ such that u and v have the same length and the same set of periods.

In this paper, we extend Guibas and Odlyzko's result to partial words with one hole. We first review, in Section 2, basic properties of words and partial words, state, in Section 3, the fundamental periodicity result of Fine and Wilf as well as its extension to partial words with one hole, state, in Section 4, the periodicity result of Guibas and Odlyzko as well as Halava et al's algorithm for computing a binary word with the same length and the same set of periods as a given word, prove, in Section 5, our main result which states that the set of all periods and the set of all local periods of a partial word with one hole are independent of the alphabet size, and describe, in Section 6, a linear time algorithm that given a partial word u with one hole, computes a partial word v with at most one hole over the alphabet $\{0, 1\}$ such that u and v have the same length, the same set of periods, and the same set of local periods.

2 Preliminaries

This section is devoted to reviewing basic concepts on words and partial words.

2.1 Words

For a detailed presentation of the matters discussed here, please refer to [6] or [11].

Let A be a nonempty finite set, or an *alphabet*. Elements of A are called *letters* and finite sequences of letters of A are called *words* over A. The unique sequence of length 0, denoted by ϵ, is called the *empty word*. The set of all words over A of finite length (greater than or equal to 0) is denoted by A^*. It is a monoid under the associative operation of concatenation or product of words (ϵ serves as identity) and is referred to as the *free monoid* generated by A. Similarly, the set of all nonempty words over A is denoted by A^+. It is a semigroup under the operation of concatenation of words and is referred to as the *free semigroup* generated by A.

A word of length n over A can be defined by a total function $u : \{1, \ldots, n\} \to A$ and is usually represented as $u = a_1a_2 \ldots a_n$ with $a_i \in A$ (the length of u or n is denoted by $|u|$). If $u = a_1 \ldots a_n$ with $a_i \in A$, then *a period of* u is a positive integer p such that $a_i = a_{i+p}$ for $1 \leq i \leq n - p$. The minimal period of u will be denoted by $p(u)$.

For a word u, the powers of u are defined inductively by $u^0 = \epsilon$ and, for any $n \geq 1$, $u^n = uu^{n-1}$. A word u is *primitive* if there exists no word v such that $u = v^n$ with $n \geq 2$.

2.2 Partial words

For a detailed presentation of the matters discussed here, please refer to [1].

A *partial word* u of length n over A is a partial function $u : \{1, \ldots, n\} \rightarrow A$. For $1 \leq i \leq n$, if $u(i)$ is defined, then we say that i belongs to the *domain* of u (denoted by $i \in D(u)$), otherwise we say that i belongs to the *set of holes* of u (denoted by $i \in H(u)$). A word over A is a partial word over A with an empty set of holes (we will sometimes refer to words as *full* words).

If u is a partial word of length n over A, then the *companion* of u (denoted by $u_\diamond$) is the total function $u_\diamond : \{1, \ldots, n\} \rightarrow A \cup \{\diamond\}$ defined by

$$u_\diamond(i) = \begin{cases} u(i) & \text{if } i \in D(u), \\ \diamond & \text{otherwise.} \end{cases}$$

The bijectivity of the map $u \mapsto u_\diamond$ allows us to define for partial words concepts such as concatenation and powers in a trivial way. The symbol $\diamond$ is viewed as a "do not know" symbol and not as a "do not care" symbol as in pattern matching. The word $u_\diamond = abb\diamond b\diamond cbb$ is the companion of the partial word u of length 9 where $D(u) = \{1, 2, 3, 5, 7, 8, 9\}$ and $H(u) = \{4, 6\}$.

A period of a partial word u over A is a positive integer p such that $u(i) = u(j)$ whenever $i, j \in D(u)$ and $i \equiv j \mod p$. In such a case, we call u *p-periodic*. Similarly, *a local period of* u is a positive integer p such that $u(i) = u(i+p)$ whenever $i, i+p \in D(u)$. In such a case, we call u *locally p-periodic*. The partial word with companion $abb\diamond bbcbb$ is locally 3-periodic but is not 3-periodic. The latter shows a difference between partial words and words since every locally p-periodic word is p-periodic. Another difference worth noting is the fact that even if the length of a partial word u is a multiple of a local period of u, then u is not necessarily a power of a shorter partial word. We will denote by $p(u)$ the minimal period of u and by $p'(u)$ the minimal local period of u. The set of all periods of u will be denoted by $\mathcal{P}(u)$ and the set of all local periods of u will be denoted by $\mathcal{P}'(u)$. Note that, for any partial word u, $\mathcal{P}(u) \neq \emptyset$, since $|u| \in \mathcal{P}(u)$ (a similar statement holds for $\mathcal{P}'(u)$).

If u and v are two partial words of equal length, then u is said to be contained in v, denoted by $u \subset v$, if $D(u) \subset D(v)$ and $u(i) = v(i)$ for all $i \in D(u)$. The order $u \subset v$ on partial words is obtained when we let $\diamond < a$ and $a \leq a$ for all $a \in A$. The partial words u and v are called *compatible*, denoted by $u \uparrow v$, if there exists a partial word w such that $u \subset w$

and $v \subset w$. We denote by $u \vee v$ the least upper bound of u and v (in other words, $u \subset u \vee v$ and $v \subset u \vee v$ and $D(u \vee v) = D(u) \cup D(v)$). As an example, $u_\diamond = aba\diamond\diamond a$ and $v_\diamond = a\diamond\diamond b\diamond a$ are the companions of two partial words u and v that are compatible and $(u \vee v)_\diamond = abab\diamond a$.

We can extend the notion of a word being primitive to a partial word being primitive as follows: A partial word u is *primitive* if there exists no word v such that $u \subset v^n$ with $n \geq 2$.

We end this section with a construction of a word of length n from a given word u of length n over the alphabet $A \cup \{\diamond\}$. Let S be a subset of $\{1, \ldots, n\}$ and $a \in A \cup \{\diamond\}$. We define the word $u(S, a)$ as follows:

$$u(S,a)(i) = \begin{cases} u(i) & \text{if } i \notin S, \\ a & \text{otherwise.} \end{cases}$$

As an example, consider the word $u = abb\diamond cbba$ over the alphabet $\{a, b, c, \diamond\}$. We can see that $u(\{1, 4, 5\}, a) = abbaabba$. If S is the singleton set $\{s\}$, then we will sometimes abbreviate $u(S, a)$ by $u(s, a)$.

3 Fine and Wilf's periodicity result

In this section, we review Fine and Wilf's periodicity result as well as its extension to partial words with one hole.

The fundamental periodicity result of Fine and Wilf can be stated as follows.

Theorem 1 (Fine and Wilf [8]) *If a word u has periods p and q and $|u| \geq p+q-\gcd(p,q)$, then u has period $\gcd(p,q)$.*

The bound $p+q-\gcd(p,q)$ turns out to be optimal, since, for example, $abaababaaba$ has periods 5 and 8, has length $11 = 5 + 8 - \gcd(5, 8) - 1$, but does not have period 1.

Berstel and Boasson proved a variant of Fine and Wilf's result for partial words with one hole.

Theorem 2 (Berstel and Boasson [1]) *If a partial word u with one hole is locally p-periodic and locally q-periodic and $|u| \geq p + q$, then u is $\gcd(p,q)$-periodic.*

The bound $p + q$ turns out to be optimal since, for example, $aaaabaaaa\diamond aa$ has one hole, is locally 5-periodic and locally 8-periodic, has length $12 = 5 + 8 - 1$, but is not 1-periodic.

Theorem 2 does not hold for two holes since, for example, $ab\diamond aba\diamond ba$ has two holes, is locally 3-periodic and locally 5-periodic, has length $\geq 3+5$, but is not 1-periodic. Note that if $\gcd(p,q)=1$, then Theorem 2 implies Theorem 1 by considering $v_\diamond = u\diamond$ or $v_\diamond = \diamond u$ where u is a word satisfying Theorem 1's assumptions.

In our recent paper [4], we extend Theorem 2 to partial words with two or three holes. The strenghtening to an arbitrary number of holes is done in our paper [2].

4 Guibas and Odlyzko's periodicity result

In the next section, we characterize the periods and local periods of partial words with one hole (see Theorem 4). This is done by extending to partial words with one hole the following result of Guibas and Odlyzko.

Theorem 3 (Guibas and Odlyzko [9]) *For every word u over an alphabet A, there exists a word v of length $|u|$ over the alphabet $\{0,1\}$ such that $\mathcal{P}(v)=\mathcal{P}(u)$.*

The proof given by Guibas and Odlyzko of Theorem 3 uses properties of correlation and is somewhat complicated. In [10], Halava et al give an elementary short constructive proof for this result. As a consequence, a linear time algorithm (Algorithm 1) is described which, given a word, computes a word over the alphabet $\{0,1\}$ with the same length and the same periods.

Halava et al's algorithm is based on the following properties of words (among others).

Lemma 1 (Halava, Harju and Ilie [10]) *Let u be a word over an alphabet A. If q is a period of u satisfying $|u| \geq p(u)+q$, then q is a multiple of $p(u)$.*

Lemma 2 (Halava, Harju and Ilie [10]) *Let u be a word over the alphabet $\{0,1\}$. Then $u0$ or $u1$ is primitive.*

Lemma 3 (Halava, Harju and Ilie [10]) *Let u be a word over an alphabet A with minimal period $p(u)$. Then there are words v,w (possibly $v=\epsilon$) and a positive integer k such that $u=(vw)^k v$, $w \neq \epsilon$ and $p(u)=|vw|$.*

Lemma 4 (Halava, Harju and Ilie [10]) *Let u be as in Lemma 3 with $k>1$, and let q be such that $|u|-p(u)<q<|u|$. Put $q=(k-1)p(u)+r$ where $|v|<r<|v|+p(u)$. Then $q\in\mathcal{P}(u)$ if and only if $r\in\mathcal{P}(vwv)$.*

We now describe Halava et al's algorithm.

Algorithm 1 (Halava, Harju and Ilie [10]) *Given as input a word u over an alphabet A, the following algorithm computes a word $Bin(u)$ of length $|u|$ over the alphabet $\{0,1\}$ such that $\mathcal{P}(Bin(u)) = \mathcal{P}(u)$.*

Find the minimal period $p(u)$ of u.

1. *If $p(u) = |u|$, then output $Bin(u) = 01^{|u|-1}$.*
2. *If $p(u) \neq |u|$, then find words satisfying Lemma 3.*
   (a) *If $k = 1$, then compute $Bin(v)$, find $c \in \{0,1\}$ such that $Bin(v)1^{|w|-1}c$ is primitive, and output $Bin(u) = Bin(v)1^{|w|-1}c\,Bin(v)$.*
   (b) *If $k > 1$, then compute $Bin(vwv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w|$ and output*

$$Bin(u) = (v'w')^k v'.$$

Note that if $u \neq \epsilon$, then Bin(u) begins with 0 (Bin(ϵ) = ϵ). We end this section with a few examples.

Example 1 1. *If $u = abbcbb$, then $Bin(u) = 011111$. Both u and $Bin(u)$ have only the period 6.*

2. *If $u = abbcbabbcba$, then $u = (a(bbcb))^2a$, $Bin((a)(bbcb)(a)) = Bin(a)1110\,Bin(a) = (0)(1110)(0)$, and $Bin(u) = 01110011100$. Both u and $Bin(u)$ have the set of periods $\{5, 10, 11\}$. Another possible value for $Bin(u)$ is 01111011110.*

5 Our main result

In this section, we extend Theorem 3 to partial words with one hole. We prove that for every partial word u with one hole over an alphabet A, there exists a partial word v of length $|u|$ with at most one hole over the alphabet $\{0,1\}$ such that $\mathcal{P}(v) = \mathcal{P}(u)$ and $\mathcal{P}'(v) = \mathcal{P}'(u)$ (Theorem 4). If $u_\diamond = a\diamond bc$, then $\mathcal{P}(u) = \mathcal{P}'(u) = \{4\}$. It is easily seen that no partial word v with one hole over $\{0,1\}$ satisfies the desired properties, but the full word 0111 does. In the sequel, $\bar{0}$ denotes 1 and $\bar{1}$ denotes 0.

Our first step in characterizing the set of periods and local periods of a partial word with one hole is to extend to partial words with one hole the properties of words and periods needed in Halava et al's proof.

The following lemma gives the structure of the set of local periods of a partial word u with one hole.

Lemma 5 *Let u be a partial word with one hole over an alphabet A. If q is a local period of u satisfying $|u| \geq p'(u) + q$, then q is a multiple of $p'(u)$.*

Proof. By Theorem 2, $\gcd(p'(u), q)$ is a period of u since $|u| \geq p'(u) + q$. Since $p(u)$ is the minimal period of u and $p'(u)$ is the minimal local period of u, we get $p'(u) \leq p(u) \leq \gcd(p'(u), q)$. We conclude that $p'(u) = \gcd(p'(u), q)$ and so $p'(u)$ divides q. □

As a consequence, if $p'(u) \leq |u|/2$, then $\mathcal{P}'(u)$ can be partitioned into two sets: the first set including $p'(u)$ and its multiples and the second set including all the local periods greater than $|u| - p'(u)$.

Lemma 6 *Let u be a partial word with one hole over the alphabet $\{0, 1\}$ which is not of the form $x \diamond x$ for any x. Then $u0$ or $u1$ is primitive.*

Proof. Assume that $u0 \subset v^k$, $u1 \subset w^\ell$ for some primitive words v, w and integers $k, \ell \geq 2$. Both $|v|$ and $|w|$ are periods of u, and, since $k, \ell \geq 2$, $|u| = k|v| - 1 = \ell|w| - 1 \geq 2\max\{|v|, |w|\} - 1 \geq |v| + |w| - 1$.

Case 1. $|u| = |v| + |w| - 1$.

Here $|v| = |w|$. Since v ends with 0 and w with 1, put $v = x0$ and $w = y1$. We get $u \subset x0x$ and $u \subset y1y$ with $|x| = |y|$. We conclude that $u = x \diamond x$ where $x = y$, a contradiction.

Case 2. $|u| > |v| + |w| - 1$.

By Theorem 2, u is also $\gcd(|v|, |w|)$-periodic. However, $\gcd(|v|, |w|)$ divides $|v|$ and $|w|$, and so $u \subset x^m$ with $|x| = \gcd(|v|, |w|)$. Since v ends with 0 and w with 1, we get that x ends with 0 and 1, a contradiction. □

Lemma 7 *Let u be a partial word with one hole over an alphabet A with minimal local period $p'(u)$. Then one of the following holds:*

1. *There are partial words $v, w_1, w_2, \ldots, w_k$ (possibly $v = \epsilon$) such that*

$$u = vw_1vw_2 \dots vw_kv,$$

where $p'(u) = |vw_1| = |vw_2| = \cdots = |vw_k|$ *and where there exists* $1 \le i \le k$ *such that* $w_i = x\diamond y$*,* $w_j = xay$ *if* $j < i$*, and* $w_j = xby$ *if* $j > i$ *for some* $a, b \in A$ *and* $x, y \in A^*$*.*

2. *There are partial words* $w, v_1, v_2, \dots, v_{k+1}$ *such that*

$$u = v_1wv_2w \dots v_kwv_{k+1},$$

where $p'(u) = |v_1w| = |v_2w| = \cdots = |v_kw| = |v_{k+1}w|$*,* $w \ne \epsilon$*,* $k \ge 1$*, and where there exists* $1 \le i \le k+1$ *such that* $v_i = x\diamond y$*,* $v_j = xay$ *if* $j < i$*, and* $v_j = xby$ *if* $j > i$ *for some* $a, b \in A$ *and* $x, y \in A^*$*.*

Proof. Let u be a partial word with one hole over A with minimal local period $p'(u)$. Then $|u| = kp'(u) + r$ where $0 \le r < p'(u)$. Put $u = v_1w_1v_2w_2 \dots v_kw_kv_{k+1}$ where $|v_1w_1| = |v_2w_2| = \cdots = |v_kw_k| = p'(u)$ and $|v_1| = |v_2| = \cdots = |v_k| = |v_{k+1}| = r$. Two cases arise.

Case 1. There exists $1 \le i \le k$ such that the hole is in w_i.

In this case, $v_1 = v_2 = \cdots = v_k = v_{k+1} = v$ for some possibly empty v. Here we get the situation described in Statement 1.

Case 2. There exists $1 \le i \le k+1$ such that the hole is in v_i.

In this case, $w_1 = w_2 = \cdots = w_k = w$ for some nonempty w (if w is empty, then $r = |v_{k+1}| = |v_k| = p'(u)$, a contradiction). Note that $k \ge 1$ (otherwise, $u = v_{k+1}$ and u has local period $|v_{k+1}| < p'(u)$ contradicting the fact that $p'(u)$ is the minimal local period of u). Here we get the situation described in Statement 2. □

Lemma 8 1. *Let* u *be as in Lemma 7(1) with* $k > 1$*, and let* q *be such that* $|u| - p'(u) < q < |u|$*. Put* $q = (k-1)p'(u) + r$ *where* $|v| < r < |v| + p'(u)$*. Also put* $H(vw_iv) = \{h\}$*. Then* $q \in \mathcal{P}(u)$ *if and only if* $q \in \mathcal{P}'(u)$*. Moreover,* $q \in \mathcal{P}'(u)$ *if and only if the following three conditions hold:*

(a) $r \in \mathcal{P}'(vw_iv)$*.*

(b) If $i \ne 1$ *and* $h + r \le |v| + p'(u)$*, then* $(vw_iv)(h+r) = a$*.*

(c) If $i \ne k$ *and* $r < h$*, then* $(vw_iv)(h-r) = b$*.*

2. *Let u be as in Lemma 7(2) with $k > 1$, and let q be such that $|u| - p'(u) < q < |u|$. Put $q = (k-1)p'(u) + r$ where $|v_i| < r < |v_i| + p'(u)$. Then $q \in \mathcal{P}(u)$ if and only if $q \in \mathcal{P}'(u)$.*

   (a) *If $i \neq k+1$ and $H(v_i) = \{h\}$, then $q \in \mathcal{P}'(u)$ if and only if the following two conditions hold:*

      i. *$r \in \mathcal{P}'(v_i w v_{i+1})$.*

      ii. *If $i \neq 1$ and $h + r \leq |v_i| + p'(u)$, then $(v_i w v_{i+1})(h + r) = a$.*

   (b) *If $i \neq 1$ and $H(v_{i-1} w v_i) = \{h\}$, then $q \in \mathcal{P}'(u)$ if and only if the following two conditions hold:*

      i. *$r \in \mathcal{P}'(v_{i-1} w v_i)$.*

      ii. *If $i \neq k+1$ and $r < h$, then $(v_{i-1} w v_i)(h - r) = b$.*

Proof. We first prove Statement 1. For any $0 < j \leq |u| - q = p'(u) + |v| - r$, we have $u_\diamond(j) = (vw_1v)_\diamond(j)$ and $u_\diamond(j+q) = (vw_kv)_\diamond(j+r)$. Hence $u(j) = u(j+q)$ if and only if $(vw_1v)(j) = (vw_kv)(j+r)$. The latter implies that $q \in \mathcal{P}'(u)$ if and only if Conditions (a)-(c) hold. To see this, first let us assume that $q \in \mathcal{P}'(u)$ and let $j, j+r \in D(vw_iv)$. We have $j \in D(vw_1v)$ and $j + r \in D(vw_kv)$ and so $j, j+q \in D(u)$. We get $u(j) = u(j+q)$ and so $(vw_iv)(j) = (vw_1v)(j) = (vw_kv)(j+r) = (vw_iv)(j+r)$ showing that Condition (a) holds. To see that Condition (b) holds, note that $h \in D(u)$ and $h + q \in D(u)$. We have $(vw_iv)(h+r) = (vw_kv)(h+r) = u(h+q) = u(h) = (vw_1v)(h) = a$. To see that Condition (c) holds, note that $h - r \in D(u)$ and $h - r + q \in D(u)$. We have $(vw_iv)(h-r) = (vw_1v)(h-r) = u(h-r) = u(h-r+q) = (vw_kv)(h) = b$.

Now, let us show that if Conditions (a)-(c) hold, then $q \in \mathcal{P}'(u)$. Let $j, j+q \in D(u)$. We get $j \in D(vw_1v)$ and $j + r \in D(vw_kv)$. If $j \notin \{h, h-r\}$, then $j \in D(vw_iv)$ and $j + r \in D(vw_iv)$. In this case, $(vw_1v)(j) = (vw_iv)(j) = (vw_iv)(j+r) = (vw_kv)(j+r)$ since Condition (a) holds, and so $u(j) = u(j+q)$. If $j = h$, then $i \neq 1$ and $j + r \in D(vw_iv)$. In this case, $u(j) = (vw_1v)(j) = (vw_iv)(j+r) = (vw_kv)(j+r) = u(j+q)$ since Condition (b) holds. If $j = h - r$, then $i \neq k$ and $j \in D(vw_iv)$. In this case, $u(j) = (vw_1v)(j) = (vw_iv)(j) = (vw_kv)(j+r) = u(j+q)$ since Condition (c) holds.

We now prove Statement 2. For any $0 < j \leq |u| - q = p'(u) + |v_i| - r$, we have $u_\diamond(j) = (v_1wv_2)_\diamond(j)$ and $u_\diamond(j+q) = (v_kwv_{k+1})_\diamond(j+r)$. Hence $u(j) = u(j+q)$ if and only if

$(v_1wv_2)(j) = (v_kwv_{k+1})(j+r)$. The proof is similar to that of Statement 1. □

Our algorithm 2, that will be described fully in Section 6, works as follows: Let A be an alphabet not containing the special symbol $\Box$. Given as input a partial word u with one hole over A where $H(u) = \{h\}$, Algorithm 2 computes a triple $T(u) = [\text{Bin}'(u), \alpha_u, \beta_u]$, where $\text{Bin}'(u)$ is a partial word of length $|u|$ over the alphabet $\{0,1\}$ such that $\text{Bin}'(u)$ does not begin with 1, $H(\text{Bin}'(u)) \subseteq \{h\}$, where $\mathcal{P}(\text{Bin}'(u)) = \mathcal{P}(u)$ and $\mathcal{P}'(\text{Bin}'(u)) = \mathcal{P}'(u)$, and where

$$\alpha_u = \begin{cases} \Box & \text{if } h - p'(u) < 1, \\ u(h - p'(u)) & \text{otherwise,} \end{cases}$$

and

$$\beta_u = \begin{cases} \Box & \text{if } h + p'(u) > |u|, \\ u(h + p'(u)) & \text{otherwise.} \end{cases}$$

In particular, $T(\diamond) = [0, \Box, \Box]$, and if $a \in A$ and $k > 1$, then $T(\diamond a^{k-1}) = [0^k, \Box, a]$. Moreover, if $\mathcal{P}(u) \neq \mathcal{P}'(u)$, then $H(\text{Bin}'(u)) = \{h\}$ and $\alpha_u = u(h - p'(u)) \neq u(h + p'(u)) = \beta_u$. Also, if $\alpha_u \neq \Box$ and $\beta_u \neq \Box$, then $H(\text{Bin}'(u)) = \{h\}$.

Lemma 9 *Let u be as in Lemma 7(1) with $k = 1$. Assume that Bin(v) begins with 0. For $c \in \{0,1\}$ such that Bin$(v)1^{|w_1|-1}c$ is primitive, $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary word $u' =$ Bin$(v)1^{|w_1|-1}c$Bin(v).*

Proof. Put $w_1 = w$. Obviously, $\mathcal{P}'(u') = \mathcal{P}(u')$, and $\mathcal{P}'(u) = \mathcal{P}(u)$ holds since every local period of u is greater than or equal to $p'(u)$. Clearly, $\mathcal{P}(u) \subseteq \mathcal{P}(u')$, since $\mathcal{P}(\text{Bin}(v)) = \mathcal{P}(v)$ and all periods q of u satisfy $q \geq p(u) \geq p'(u) = |vw| = |\text{Bin}(v)1^{|w|-1}c|$. Assume then that there exists $q \in \mathcal{P}(u') \setminus \mathcal{P}(u)$ and also that q is minimal with this property. Either $q < |\text{Bin}(v)|$ or $|\text{Bin}(v)| + |w| - 1 \leq q < |u|$, since $\text{Bin}(v)$ does not begin with 1.

If $q < |\text{Bin}(v)|$, then, by the minimality of q, q is the minimal period of u', and Lemma 1 implies that $p'(u)$ is a multiple of q, and so $\text{Bin}(v)1^{|w|-1}c$ is not primitive, a contradiction. If $q = |\text{Bin}(v)| + |w| - 1$, then $c = 0$. In this case, if $|w| > 1$, we get $\text{Bin}(v)1 = 0\text{Bin}(v)$, which is impossible, and if $|w| = 1$, we get that $\text{Bin}(v)$ consists of 0's only, that 1 is a period of v and hence of $u = v\diamond v$, and that $1 = p'(u) = |v\diamond|$ and so $v = \epsilon$ and $\text{Bin}(v) = \epsilon$, a contradiction.

Therefore $q > |\mathrm{Bin}(v)| + |w| - 1$, and $q > p'(u) = |vw|$ since $p'(u) \notin \mathcal{P}(u') \setminus \mathcal{P}(u)$. Put $q = p'(u) + r$ where $r > 0$. Then r is a period of $\mathrm{Bin}(v)$ and hence of v. But this implies $q \in \mathcal{P}(u)$, a contradiction. □

Lemma 10 *Let u be as in Lemma 7(2) with $k = 1$. Assume that $v_1 = x\diamond y$ and $v_2 = xby$. Assume that $T(v_1) = [Bin'(v_1), \alpha, \beta]$ with $H(Bin'(v_1)) \subseteq H(v_1) = \{h\}$.*

1. *If $\beta = \square$, then let $c \in \{0, 1\}$ be such that $Bin(v_2)1^{|w|-1}c$ is primitive. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary word*

$$u' = Bin(v_2)1^{|w|-1}c\,Bin(v_2).$$

2. *If $\beta \neq \square$, then define d as follows:*

$$d = \begin{cases} Bin'(v_1)(h - p'(v_1)) & \text{if } \alpha \neq \square \text{ and } b = \alpha, \\ \overline{Bin'(v_1)(h - p'(v_1))} & \text{if } \alpha \neq \square \text{ and } b \neq \alpha, \\ 1 & \text{otherwise.} \end{cases}$$

   (a) *If $Bin'(v_1) = 0^{|x|}\diamond 1^{|y|}$, then let $c = 1$.*

   (b) *Otherwise, if $Bin'(v_1)1^{|w|-1}$ is not of the form $z\diamond z$ for any z, then let $c \in \{0, 1\}$ be such that $Bin'(v_1)1^{|w|-1}c$ is primitive.*

   (c) *Otherwise, if $Bin'(v_1)1^{|w|-1}$ is of the form $z\diamond z$ for some z, then let $c = \bar{d}$.*

   Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary partial word

$$u' = Bin'(v_1)1^{|w|-1}c\,Bin'(v_1)(H(Bin'(v_1)), d).$$

Proof. We first prove Statement 1. There exists $c \in \{0, 1\}$ such that $\mathrm{Bin}(v_2)1^{|w|-1}c$ is primitive by Lemma 2. The equality $\mathcal{P}'(u) = \mathcal{P}(u)$ holds since every local period of u is greater than or equal to $p'(u)$, and the equality $\mathcal{P}'(u') = \mathcal{P}(u')$ holds trivially.

To see that $\mathcal{P}(u) \subseteq \mathcal{P}(u')$, first note that $\mathcal{P}(\mathrm{Bin}(v_2)) = \mathcal{P}(v_2)$, and all periods q of u satisfy $q \geq p'(u)$. If $q = p'(u)$, then q is a period of u'. If $q > p'(u)$, put $q = p'(u) + r$ where $r > 0$. Then r is a local period of v_1. If $\beta = \square$, then $h + r \geq h + p'(v_1) > |v_1|$. In this case, r is a period of v_2 and hence of $\mathrm{Bin}(v_2)$, and so $q \in \mathcal{P}(u')$.

Assume then that there exists $q \in \mathcal{P}(u') \setminus \mathcal{P}(u)$ and also that q is minimal with this property. Either $q < |\text{Bin}(v_2)|$ or $|\text{Bin}(v_2)| + |w| - 1 \leq q < |u|$, since $\text{Bin}(v_2)$ does not begin with 1. If $q < |\text{Bin}(v_2)|$, then, by the minimality of q, q is the minimal period of u', and Lemma 1 implies that $p'(u)$ is a multiple of q, and so $\text{Bin}(v_2)1^{|w|-1}c$ is not primitive, a contradiction. If $q = |\text{Bin}(v_2)| + |w| - 1$, then $c = 0$. In this case, if $|w| > 1$, we get $\text{Bin}(v_2)1 = 0\text{Bin}(v_2)$, which is impossible, and if $|w| = 1$, we get that $\text{Bin}(v_2)$ consists of 0's only and therefore $\text{Bin}(v_2)1^{|w|-1}c = \text{Bin}(v_2)0$ is not primitive. Hence $q > |\text{Bin}(v_2)| + |w| - 1$, and $q > p'(u)$ since $p'(u) \notin \mathcal{P}(u') \setminus \mathcal{P}(u)$. By putting $q = p'(u) + r$ where $r > 0$, we get that r is a period of $\text{Bin}(v_2)$ and hence of v_2. Therefore $q \in \mathcal{P}(u)$.

We now prove Statement 2 when $\text{Bin}'(v_1)$ has a hole (when $\text{Bin}'(v_1)$ is full, we have that $\alpha = \square$ and the proof is simpler). We first prove Statements 2(a) and 2(b) when $\text{Bin}'(v_1)$ begins with 0. Note that in the case of Statement 2(a) where $\text{Bin}'(v_1) = 0^{|x|}\diamond 1^{|y|}$, we have that $\text{Bin}'(v_1)1^{|w|-1}c = 0^{|x|}\diamond 1^{|y|+|w|}$ is primitive. In the case of Statement 2(b), Lemma 6 implies that there exists $c \in \{0, 1\}$ such that $\text{Bin}'(v_1)1^{|w|-1}c$ is primitive since $\text{Bin}'(v_1)1^{|w|-1}$ is not of the form $z\diamond z$.

Now, the equality $\mathcal{P}'(u) = \mathcal{P}(u)$ holds since every local period of u is greater than or equal to $p'(u)$. To see that $\mathcal{P}'(u') = \mathcal{P}(u')$, first we note that the inclusion $\mathcal{P}(u') \subseteq \mathcal{P}'(u')$ clearly holds. So let $q \in \mathcal{P}'(u')$. If $q \leq |u| - p'(u)$, then $|u'| = |u| \geq p'(u) + q \geq p'(u') + q$ and by Theorem 2, $\gcd(p'(u'), q) \in \mathcal{P}(u')$ and so $\gcd(p'(u'), q) \in \mathcal{P}'(u')$. By the minimality of $p'(u')$, we have $\gcd(p'(u'), q) = p'(u')$, and therefore $p'(u')$ divides q, which implies $q \in \mathcal{P}(u')$ since $p'(u') \in \mathcal{P}(u')$. If $q > |u| - p'(u)$, then clearly $q \in \mathcal{P}(u')$.

To see that $\mathcal{P}(u) \subseteq \mathcal{P}(u')$, first note that $\mathcal{P}(\text{Bin}'(v_1)) = \mathcal{P}(v_1)$, $\mathcal{P}'(\text{Bin}'(v_1)) = \mathcal{P}'(v_1)$, and all periods q of u satisfy $q \geq p'(u)$. If $q = p'(u)$, then q is a period of u'. If $q > p'(u)$, put $q = p'(u) + r$ where $r > 0$.

- If $r \geq h$, then r is a period of v_1 and hence of $\text{Bin}'(v_1)$. In this case, $q \in \mathcal{P}(u')$.
- If $r < h$, then r is a local period of v_1 and hence of $\text{Bin}'(v_1)$. Here $\alpha \neq \square$ since $p'(v_1) \leq r < h$ and $|v_1| > p'(v_1) + r$ since $\beta \neq \square$. By Lemma 5, r is a multiple of $p'(v_1)$. We have $\alpha = v_1(h - p'(v_1)) = v_1(h - r) = v_2(h) = b$ and $\text{Bin}'(v_1)(h - r) = \text{Bin}'(v_1)(h - p'(v_1)) = d = \text{Bin}'(v_1)(h, d)(h)$, and so $q \in \mathcal{P}(u')$.

Assume then that there exists $q \in \mathcal{P}(u') \setminus \mathcal{P}(u)$ and also that q is minimal with this property. Either $q < |\text{Bin}'(v_1)|$ or $|\text{Bin}'(v_1)| + |w| - 1 \leq q < |u|$, since $\text{Bin}'(v_1)$ does not begin with 1 or $\diamond$. If $q < |\text{Bin}'(v_1)|$, then, by the minimality of q, q is the minimal period of u', and Lemma 5 implies that $p'(u)$ is a multiple of q, and so $\text{Bin}'(v_1)1^{|w|-1}c$ is not primitive, a contradiction. If $q = |\text{Bin}'(v_1)| + |w| - 1$, then $c = 0$. In this case, if $|w| > 1$ and $d = 0$, we get $\text{Bin}'(v_1)1 = 0\text{Bin}'(v_1)(h, 0)$, which is impossible. If $|w| > 1$ and $d = 1$, we get that $\text{Bin}'(v_1)$ looks like $0^{|x|}\diamond 1^{|y|}$ and therefore that $c = 1$, a contradiction. If $|w| = 1$ and $d = 0$, we get that $\text{Bin}'(v_1)$ consists of 0's only and therefore that $\text{Bin}'(v_1)1^{|w|-1}c = \text{Bin}'(v_1)0$ is not primitive. And if $|w| = 1$ and $d = 1$, we get an impossible situation. Therefore $q > |\text{Bin}'(v_1)| + |w| - 1$, and $q > p'(u)$ since $p'(u) \notin \mathcal{P}(u') \setminus \mathcal{P}(u)$. By putting $q = p'(u) + r$ where $r > 0$, we get that r is a local period of $\text{Bin}'(v_1)$ and hence of v_1. If $r \geq h$, then $q \in \mathcal{P}(u)$. If $r < h$, then $\alpha \neq \square$ and $|v_1| > p'(v_1) + r$. By Lemma 5, r is a multiple of $p'(v_1)$. We have $\text{Bin}'(v_1)(h - r) = \text{Bin}'(v_1)(h, d)(h)$ and so $d = \text{Bin}'(v_1)(h - p'(v_1))$. Then $b = \alpha$ and so $v_1(h - r) = v_1(h - p'(v_1)) = \alpha = b = v_2(h)$ and $q \in \mathcal{P}(u)$.

We now prove Statements 2(a) and 2(b) and 2(c) when $\text{Bin}'(v_1)$ begins with a hole (here $\alpha = \square$). The case where $\text{Bin}'(v_1) = \diamond 1^{|y|}$ is impossible here. There exists $c \in \{0, 1\}$ such that $\text{Bin}'(v_1)1^{|w|-1}c$ is primitive since $\text{Bin}'(v_1)1^{|w|-1}$ cannot be of the form $z\diamond z$ (otherwise $z = \epsilon$ and $v_1 = \diamond$ and $\text{Bin}'(v_1) = 0$). The equalities $\mathcal{P}'(u) = \mathcal{P}(u)$ and $\mathcal{P}'(u') = \mathcal{P}(u')$ hold as above, as well as the inclusion $\mathcal{P}(u) \subseteq \mathcal{P}(u')$. Assume then that there exists $q \in \mathcal{P}(u') \setminus \mathcal{P}(u)$ and also that q is minimal with this property. The cases $q < |\text{Bin}'(v_1)|$ and $|\text{Bin}'(v_1)| + |w| - 1 < q < |u|$ follow as above. If $q = |\text{Bin}'(v_1)| + |w| - 1$, then we consider the following cases: If $|w| > 1$, we get that $\text{Bin}'(v_1)$ looks like $\diamond 1^{|y|}$ which is an impossible situation (the same is true when $|w| = 1$ and $c = 1$). And if $|w| = 1$ and $c = 0$, we get an impossible situation. If $|\text{Bin}'(v_1)| \leq q < |\text{Bin}'(v_1)| + |w| - 1$ and $c = 1$, then we get that $\text{Bin}'(v_1)$ looks like $\diamond 1^{|y|}$ which is impossible. And if $|\text{Bin}'(v_1)| \leq q < |\text{Bin}'(v_1)| + |w| - 1$ and $c = 0$, then we get an impossible situation or we get that $\text{Bin}'(v_1)1^{|w|-1}c$ is not primitive.

We are left to prove Statement 2(c) when $\text{Bin}'(v_1)$ begins with 0 (or when z begins with 0). $\text{Bin}'(v_1)1^{|w|-1}$ is of the form $z\diamond z$, and consequently $\text{Bin}'(v_1) = z_1 1^{|w|-1}\diamond z_1$ for some z_1 that begins with 0. If $p'(v_1) = |z_1 w|$, then v_1 satisfies Lemma 7(1) and our algorithm (based on Lemma 9 in this case) returns a full binary word, a contradiction. If $|w| > 1$ and $|z_1| \leq p'(v_1) \leq |z_1| + |w| - 1$, then z_1 begins with 1, a contradiction. If $|w| = 1$ and $p'(v_1) = |z_1|$, then $\text{Bin}'(v_1) = 0\diamond 0$ and $\alpha \neq \square$. Put $v_1 = e\diamond e$ for some letter e. We have $u = v_1 w v_2 = e\diamond ewebe$ and since $p'(u) = |v_1 w| = 4$, we get $b \neq w$ and ($e \neq b$ or $e \neq w$). If

$b = \alpha$, then $c = 1$ and $d = 0$ and $u' = 0\diamond01000$ satisfies the required properties. If $b \neq \alpha$, then $c = 0$ and $d = 1$ and $u' = 0\diamond00010$ satisfies the required properties. If $p'(v_1) < |z_1|$, then we argue as follows. Here both $p'(v_1)$ and $|z_1| + |w|$ are local periods of v_1. Since $|v_1| \geq p'(v_1) + |z_1| + |w|$, Theorem 2 says that $\gcd(p'(v_1), |z_1| + |w|) \in \mathcal{P}(v_1)$. But since $p'(v_1)$ is the minimal local period of v_1, we get that $\gcd(p'(v_1), |z_1| + |w|) = p'(v_1)$ and so $p'(v_1)$ divides $|z_1| + |w|$. Therefore $x\diamond$ can be written as $(x_1e)^n x_1\diamond$ where $n > 0$, $p'(v_1) = |x_1e|$, e is a letter, and x_1 is a word. We consider the case where $|w| = 1$ and then the case where $|w| > 1$. If $|w| = 1$, then $u = (x_1e)^n x_1\diamond(x_1e)^n x_1 w(x_1e)^n x_1 b(x_1e)^n x_1$. Since $p'(u) = |v_1w|$, we get $b \neq w$ and ($e \neq b$ or $e \neq w$). If $b = \alpha$, then $u' = (z_2d)^n z_2\diamond(z_2d)^n z_2\bar{d}(z_2d)^n z_2 d(z_2d)^n z_2$ for some z_2. And if $b \neq \alpha$, then $u' = (z_2\bar{d})^n z_2\diamond(z_2\bar{d})^n z_2\bar{d}(z_2\bar{d})^n z_2 d(z_2\bar{d})^n z_2$ for some z_2. In either case, u' satisfies the required properties. If $|w| > 1$, then $|x_1| \geq |w|$. Put $w = w_1f$ and $x_1 = x_2w_2$ where f is a letter, x_2 is a nonempty word, and w_1, w_2 are words of length $|w| - 1$. Here $u = (x_2w_2e)^n x_2w_2\diamond(x_2w_2e)^n x_2w_1f(x_2w_2e)^n x_2w_2b(x_2w_2e)^n x_2$. Since $p'(u) = |v_1w|$, we get $w_1f \neq w_2b$ and ($e \neq b$ or $w_1f \neq w_2e$). If $b = \alpha$, then

$$u' = (z_2 1^{|w|-1} d)^n z_2 1^{|w|-1}\diamond(z_2 1^{|w|-1} d)^n z_2 1^{|w|-1}\bar{d}(z_2 1^{|w|-1} d)^n z_2 1^{|w|-1} d(z_2 1^{|w|-1} d)^n z_2$$

for some z_2. And if $b \neq \alpha$, then

$$u' = (z_2 1^{|w|-1}\bar{d})^n z_2 1^{|w|-1}\diamond(z_2 1^{|w|-1}\bar{d})^n z_2 1^{|w|-1}\bar{d}(z_2 1^{|w|-1}\bar{d})^n z_2 1^{|w|-1} d(z_2 1^{|w|-1}\bar{d})^n z_2$$

for some z_2. In either case, u' satisfies the required properties. ☐

Lemma 11 *Let u be as in Lemma 7(2) with $k = 1$. Assume that $v_1 = xay$ and $v_2 = x\diamond y$. Assume that $T(v_2) = [Bin'(v_2), \alpha, \beta]$ with $H(Bin'(v_2)) \subseteq H(v_2) = \{h\}$.*

1. *If $\alpha = \square$, then let $c \in \{0, 1\}$ be such that $Bin(v_1)1^{|w|-1}c$ is primitive. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary word*

$$u' = Bin(v_1)1^{|w|-1}cBin(v_1).$$

2. *If $\alpha \neq \square$, then define d as follows:*

$$d = \begin{cases} Bin'(v_2)(h + p'(v_2)) & \text{if } \beta \neq \square \text{ and } a = \beta, \\ \overline{Bin'(v_2)(h + p'(v_2))} & \text{if } \beta \neq \square \text{ and } a \neq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $c \in \{0,1\}$ be such that $Bin'(v_2)(H(Bin'(v_2)), d)1^{|w|-1}c$ is primitive (let $c = 1$ in the case where $Bin'(v_2) = 0^{|x|}\diamond 1^{|y|}$). Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary partial word

$$u' = Bin'(v_2)(H(Bin'(v_2)), d)1^{|w|-1}cBin'(v_2).$$

Proof. We first prove Statement 1. There exists $c \in \{0,1\}$ such that $\mathrm{Bin}(v_1)1^{|w|-1}c$ is primitive by Lemma 2. The equality $\mathcal{P}'(u) = \mathcal{P}(u)$ holds since every local period of u is greater than or equal to $p'(u)$, and the equality $\mathcal{P}'(u') = \mathcal{P}(u')$ holds trivially.

To see that $\mathcal{P}(u) \subseteq \mathcal{P}(u')$, first note that $\mathcal{P}(\mathrm{Bin}(v_1)) = \mathcal{P}(v_1)$, and all periods q of u satisfy $q \geq p'(u)$. If $q = p'(u)$, then q is a period of u'. If $q > p'(u)$, put $q = p'(u) + r$ where $r > 0$. Then r is a local period of v_2. If $\alpha = \square$, then $r \geq p'(v_2) \geq h$. In this case, r is a period of v_1 and hence of $\mathrm{Bin}(v_1)$, and so $q \in \mathcal{P}(u')$.

Assume then that there exists $q \in \mathcal{P}(u') \setminus \mathcal{P}(u)$ and also that q is minimal with this property. Either $q < |\mathrm{Bin}(v_1)|$ or $|\mathrm{Bin}(v_1)| + |w| - 1 \leq q < |u|$, since $\mathrm{Bin}(v_1)$ does not begin with 1. If $q < |\mathrm{Bin}(v_1)|$, then, by the minimality of q, q is the minimal period of u', and Lemma 1 implies that $p'(u)$ is a multiple of q, and so $\mathrm{Bin}(v_1)1^{|w|-1}c$ is not primitive, a contradiction. If $q = |\mathrm{Bin}(v_1)| + |w| - 1$, then $c = 0$. In this case, if $|w| > 1$, we get $\mathrm{Bin}(v_1)1 = 0\mathrm{Bin}(v_1)$, which is impossible, and if $|w| = 1$, we get that $\mathrm{Bin}(v_1)$ consists of 0's only and therefore $\mathrm{Bin}(v_1)1^{|w|-1}c = \mathrm{Bin}(v_1)0$ is not primitive. Hence $q > |\mathrm{Bin}(v_1)| + |w| - 1$, and $q > p'(u)$ since $p'(u) \notin \mathcal{P}(u') \setminus \mathcal{P}(u)$. By putting $q = p'(u) + r$ where $r > 0$, we get that r is a period of $\mathrm{Bin}(v_1)$ and hence of v_1. Therefore $q \in \mathcal{P}(u)$.

We now prove Statement 2 when $\mathrm{Bin}'(v_2)$ has a hole (when $\mathrm{Bin}'(v_2)$ is full, we have that $\beta = \square$ and the proof is simpler). Note that $\mathrm{Bin}'(v_2)(h, d)$ begins with 0 (otherwise, $h = 1$ and $\alpha = \square$). Note also that in the case where $\mathrm{Bin}'(v_2) = 0^{|x|}\diamond 1^{|y|}$, we have that $\mathrm{Bin}'(v_2)(h, d)1^{|w|-1}c = 0^{|x|}d1^{|y|+|w|}$ is primitive.

As in the proof of Lemma 10, $\mathcal{P}'(u) = \mathcal{P}(u)$ and $\mathcal{P}'(u') = \mathcal{P}(u')$. To see that $\mathcal{P}(u) \subseteq \mathcal{P}(u')$, first note that all periods q of u satisfy $q \geq p'(u) = |v_1 w| = |\mathrm{Bin}'(v_2)(h, d)1^{|w|-1}c|$. Clearly $p'(u) \in \mathcal{P}(u)$ and $p'(u) \in \mathcal{P}(u')$. So put $q = p'(u) + r$ with $r > 0$. We get that r is a local period of v_2 and hence r is a local period of $\mathrm{Bin}'(v_2)$. We consider the case where $h + r > |v_2|$, and then the case where $h + r \leq |v_2|$. If $h + r > |v_2|$, then $q \in \mathcal{P}(u')$. If $h + r \leq |v_2|$, then $\beta \neq \square$ since $h + p'(v_2) \leq h + r \leq |v_2|$ and $|v_2| > p'(v_2) + r$

since $\alpha \neq \Box$. By Lemma 5, r is a multiple of $p'(v_2)$. We have $v_2(h+r) = v_1(h)$ and so $\beta = v_2(h + p'(v_2)) = v_2(h+r) = v_1(h) = a$. In this case, $d = \text{Bin}'(v_2)(h + p'(v_2))$ and so $\text{Bin}'(v_2)(h+r) = \text{Bin}'(v_2)(h,d)(h)$ implying $q \in \mathcal{P}(u')$.

To see that $\mathcal{P}(u') \subseteq \mathcal{P}(u)$, assume that there exists $q \in \mathcal{P}(u') \setminus \mathcal{P}(u)$ and also that q is minimal with this property. Either $q < |v_1|$ or $|v_1| + |w| - 1 \leq q < |u|$, since $\text{Bin}'(v_2)(h,d)$ does not begin with 1. If $q < |v_1|$, then, by the minimality of q, q is the minimal period of u', and Lemma 5 implies that $p'(u)$ is a multiple of q, and so we get a contradiction with the choice of c. If $q = |v_1| + |w| - 1$, then $c = 0$. In this case, if $|w| > 1$ and $d = 1$, we get $\text{Bin}'(v_2)(h,1)1 = 0\text{Bin}'(v_2)$, which is impossible. If $|w| > 1$ and $d = 0$, we get that $\text{Bin}'(v_2)$ looks like $0^{|x|}\diamond 1^{|y|}$ and therefore that $c = 1$, a contradiction. If $|w| = 1$ and $d = 1$, we get an impossible situation. And if $|w| = 1$ and $d = 0$, we get that $\text{Bin}'(v_2)(h,0)$ consists of 0's only and therefore that $\text{Bin}'(v_2)(h,0)1^{|w|-1}c = \text{Bin}'(v_2)(h,0)0$ is not primitive. Therefore $q > |v_1| + |w| - 1$, and $q > p'(u)$ since $p'(u) \notin \mathcal{P}(u') \setminus \mathcal{P}(u)$. Put $q = p'(u) + r$ where $r > 0$. We get that r is a local period of $\text{Bin}'(v_2)$ and hence of v_2. If $h + r > |v_2|$, then $q \in \mathcal{P}(u)$. If $h + r \leq |v_2|$, then $\beta \neq \Box$ and $|v_2| > p'(v_2) + r$. By Lemma 5, r is a multiple of $p'(v_2)$. We get $\text{Bin}'(v_2)(h+r) = \text{Bin}'(v_2)(h,d)(h)$ and so $d = \text{Bin}'(v_2)(h+r) = \text{Bin}'(v_2)(h + p'(v_2))$. In this case, $a = \beta$ and so $v_1(h) = a = \beta = v_2(h + p'(v_2)) = v_2(h+r)$ implying $q \in \mathcal{P}(u)$. $\Box$

Lemma 12 *Let u be as in Lemma 7(1) with $k > 1$. Assume that $T(vw_iv) = [Bin'(vw_iv), \alpha, \beta]$ with $H(Bin'(vw_iv)) \subseteq H(vw_iv) = \{h\}$.*

1. *Assume that $i = 1$.*

   (a) *If $\beta = \Box$, then put $Bin(vw_kv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_k|$. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary word*

   $$u' = (v'w')^k v'.$$

   (b) *If $\alpha = \Box$ and $\beta \neq \Box$, then put $Bin'(vw_iv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_i|$. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary partial word*

   $$u' = (v'w')(h, \diamond)((v'w')(h, \bar{d}))^{k-1} v'$$

   where d is defined as follows:

   $$d = \begin{cases} 1 & \text{if } v = \epsilon \text{ and } x \neq \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

(c) If $\alpha \neq \square$ and $\beta \neq \square$, then put $Bin'(vw_iv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_i|$. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary partial word

$$u' = v'w'((v'w')(h, \bar{d}))^{k-1}v'$$

where d is defined as follows:

$$d = \begin{cases} \overline{Bin'(vw_iv)(h - p'(vw_iv))} & \text{if } b = \alpha, \\ Bin'(vw_iv)(h - p'(vw_iv)) & \text{if } b \neq \alpha. \end{cases}$$

2. Assume that $i = k$.

(a) If $\alpha = \square$, then put $Bin(vw_1v) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_1|$. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary word

$$u' = (v'w')^k v'.$$

(b) If $\alpha \neq \square$ and $\beta = \square$, then put $Bin'(vw_iv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_i|$. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary partial word

$$u' = ((v'w')(h, d))^{k-1}(v'w')(h, \diamond)v'$$

where d is defined as follows:

$$d = \begin{cases} 1 & \text{if } v = \epsilon \text{ and } y \neq \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

(c) If $\alpha \neq \square$ and $\beta \neq \square$, then put $Bin'(vw_iv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_i|$. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary partial word

$$u' = ((v'w')(h, d))^{k-1}v'w'v'$$

where d is defined as follows:

$$d = \begin{cases} Bin'(vw_iv)(h + p'(vw_iv)) & \text{if } a = \beta, \\ \overline{Bin'(vw_iv)(h + p'(vw_iv))} & \text{if } a \neq \beta. \end{cases}$$

3. Assume that $1 < i < k$ and $a = b$. Put $Bin(vw_1v) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_1|$. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary partial word

$$u' = (v'w')^{i-1}(v'w')(h, \diamond)(v'w')^{k-i}v'.$$

4. *Assume that* $1 < i < k$ *and* $a \neq b$.

(a) *If* $\alpha \neq \square$ *and* $\beta = \square$, *then put* $Bin(vw_kv) = v'w'v'$ *where* $|v'| = |v|$ *and* $|w'| = |w_k|$, *and put* $d = \overline{(v'w')(h)}$. *Then* $\mathcal{P}(u') = \mathcal{P}(u)$ *and* $\mathcal{P}'(u') = \mathcal{P}'(u)$, *for the binary partial word*

$$u' = ((v'w')(h,d))^{i-1}(v'w')(h,\diamond)((v'w')(h,\bar{d}))^{k-i}v'.$$

(b) *If* $\beta \neq \square$ *and* $x = \epsilon$, *then put* $Bin(vw_1v) = v'w'v'$ *where* $|v'| = |v|$ *and* $|w'| = |w_1|$, *and put* $d = (v'w')(h)$. *Then* $\mathcal{P}(u') = \mathcal{P}(u)$ *and* $\mathcal{P}'(u') = \mathcal{P}'(u)$, *for the binary partial word*

$$u' = ((v'w')(h,d))^{i-1}(v'w')(h,\diamond)((v'w')(h,\bar{d}))^{k-i}v'.$$

(c) *If* ($\alpha = \square$ *and* $\beta = \square$) *or* ($\beta \neq \square$ *and* $x \neq \epsilon$), *then put* $Bin'(vw_iv) = v'w'v'$ *where* $|v'| = |v|$ *and* $|w'| = |w_i|$. *Then* $\mathcal{P}(u') = \mathcal{P}(u)$ *and* $\mathcal{P}'(u') = \mathcal{P}'(u)$, *for the binary partial word*

$$u' = ((v'w')(h,d))^{i-1}(v'w')(h,\diamond)((v'w')(h,\bar{d}))^{k-i}v'$$

where d *is defined as follows:*

$$d = \begin{cases} \overline{Bin'(vw_iv)(h - p'(vw_iv))} & \text{if } \alpha \neq \square \text{ and } b = \alpha \text{ and } a = \beta, \\ Bin'(vw_iv)(h - p'(vw_iv)) & \text{if } \alpha \neq \square \text{ and } (b \neq \alpha \text{ or } a \neq \beta), \\ Bin'(vw_iv)(h + p'(vw_iv)) & \text{if } \alpha = \square \text{ and } \beta \neq \square \text{ and } a = \beta, \\ \overline{Bin'(vw_iv)(h + p'(vw_iv))} & \text{if } \alpha = \square \text{ and } \beta \neq \square \text{ and } a \neq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First, let us show that $\mathcal{P}'(u) = \mathcal{P}(u)$ for Statements 1, 2 and 3. The inclusion $\mathcal{P}(u) \subseteq \mathcal{P}'(u)$ clearly holds. So let $q \in \mathcal{P}'(u)$. If $q \leq |u| - p'(u)$, then q is a multiple of $p'(u)$ by Lemma 5. In this case, since $p'(u) \in \mathcal{P}(u)$, also $q \in \mathcal{P}(u)$. If $q > |u| - p'(u)$, then clearly $q \in \mathcal{P}(u)$.

Now, let us show that $\mathcal{P}(u') = \mathcal{P}(u)$. Obviously, $|u| \in \mathcal{P}(u)$ and $|u| \in \mathcal{P}(u')$.

First, consider q with $q \leq |u| - p'(u)$. If $q \in \mathcal{P}(u)$, then Lemma 5 gives that q is a multiple of $p'(u)$, and therefore $q \in \{p'(u), 2p'(u), \ldots, (k-1)p'(u)\}$. For Statements 1, 2 and 3 we get $q \in \mathcal{P}(u')$ since $p'(u) \in \mathcal{P}(u')$, and for Statement 4 it is impossible. On the other hand, assume that $q \in \mathcal{P}(u')$. Now, $|u'| = |u| \geq p'(u) + q$, and thus, by Theorem 1 or Theorem 2, $\gcd(p'(u), q) \in \mathcal{P}(u')$. For Statement 1(a), since $p'(u) = |v'w'|$ is a multiple of $\gcd(p'(u), q)$, we get that $\gcd(p'(u), q)$ is a period of $v'w'v'$ and hence of vw_kv. So $\gcd(p'(u), q) \in \mathcal{P}(u)$ and since q is a multiple of $\gcd(p'(u), q)$, we also get $q \in \mathcal{P}(u)$. For Statement 1(c), we have that $v'w'v'$ has a hole and that $\gcd(p'(u), q)$ is a period of $(v'w')(h, \bar{d})v'$. We get that $\gcd(p'(u), q)$ is a period of vw_kv and $q \in \mathcal{P}(u)$ as above. Statement 2 is handled similarly as Statement 1, and Statement 3 as Statement 2(a). For Statement 4, since $p'(u) = |v'w'|$ is a multiple of $\gcd(p'(u), q)$, we get that $\gcd(p'(u), q)$ is a period of both $(v'w')(h, d)v'$ and $(v'w')(h, \bar{d})v'$. We get $\gcd(p'(u), q) = p'(u)$ which is impossible.

Second, consider q with $|u| - p'(u) < q < |u|$, and put $q = (k-1)p'(u) + r$ where $|v| < r < p'(u) + |v|$. For Statement 3, $q \in \mathcal{P}(u)$ if and only if $r \in \mathcal{P}(vw_1v)$ if and only if $r \in \mathcal{P}(v'w'v')$ if and only if $q \in \mathcal{P}(u')$.

We now prove that $q \in \mathcal{P}(u)$ if and only if $q \in \mathcal{P}(u')$ for Statement 1 (this is proved similarly for Statement 2). For Statement 1(a), $q \in \mathcal{P}(u)$ if and only if $r \in \mathcal{P}(vw_kv)$ if and only if $r \in \mathcal{P}(v'w'v')$ if and only if $q \in \mathcal{P}(u')$. For Statements 1(b) and 1(c), if $q \in \mathcal{P}(u)$, then the conditions of Lemma 8(1)(a)(c) hold. Here $r \in \mathcal{P}'(vw_iv)$ by Lemma 8(1)(a). We consider the following two cases:

Case 1. $r < h$.

We have $|v| < r < h$ and so $x \neq \epsilon$ (otherwise $|vx| < r < h$ which is impossible since $h = |vx| + 1$). Here $\alpha \neq \square$ since $p'(vw_iv) \leq r < h$. Here $|vw_iv| \geq p'(vw_iv) + r$ (otherwise $|vw_iv| < p'(vw_iv| + r < p'(vw_iv) + h$ and $\beta = \square$, a contradiction). By Lemma 5, r is a multiple of $p'(vw_iv)$. Since $i \neq k$, we have $(vw_iv)(h - r) = b$ by Lemma 8(1)(c) and so $b = (vw_iv)(h-r) = (vw_iv)(h-p'(vw_iv)) = \alpha$. For Statement 1(c), we get $(v'w'v')(h-r) = \bar{d}$ since $d = \overline{(v'w'v')(h - p'(vw_iv))}$ and $q \in \mathcal{P}(u')$ by Lemma 8(1).

Case 2. $r \geq h$.

For Statement 1(b), we have $r \in \mathcal{P}(v'w'v')$. We conclude that $r \in \mathcal{P}'((v'w')(h, \diamond)v')$ and $q \in \mathcal{P}(u')$. For Statement 1(c), we have $r \in \mathcal{P}'(v'w'v')$ and the result follows by Lemma 8(1).

The cases $r \geq h$ and $r < h$ are handled similarly as above in order to show that if $q \in \mathcal{P}(u')$ then $q \in \mathcal{P}(u)$. Note that for Statement 1(b), we have that $r \in \mathcal{P}'((v'w')(h, \diamond)v')$. If $v'w'v'$ has a hole, we get $r \in \mathcal{P}'(v'w'v') = \mathcal{P}'(vw_iv)$. If $v'w'v'$ is full, then $r \in \mathcal{P}((v'w')(h, d)v')$. So $r \in \mathcal{P}(v'w'v')$ by the definition of d. Hence $r \in \mathcal{P}'(vw_iv)$. For Statement 1(c), we have that $r \in \mathcal{P}'(v'w'v') = \mathcal{P}'(vw_iv)$.

For Statement 4, we first show that if $q \in \mathcal{P}(u)$ then $q \in \mathcal{P}(u')$. If $q \in \mathcal{P}(u)$, then the three conditions of Lemma 8(1) hold. Here $r \in \mathcal{P}'(vw_iv)$ by Lemma 8(1)(a). We consider the following six cases:

Case 3. $h + r > |vw_iv|$ and $r \geq h$.

For Statement 4(a), r is a period of vw_kv and hence of $\text{Bin}(vw_kv) = v'w'v'$, and so $r \in \mathcal{P}'((v'w')(h, \diamond)v')$ and $q \in \mathcal{P}(u')$. For Statement 4(b), r is a period of vw_1v and the result follows similarly. For Statement 4(c), we get $r \in \mathcal{P}'((v'w')(h, \diamond)v')$ since $r \in \mathcal{P}'(v'w'v')$.

Case 4. $h + r \leq |vw_iv|$ and $r \geq h$ and $|vw_iv| \geq p'(vw_iv) + r$.

Here $|vw_iv| \geq p'(vw_iv) + r \geq p'(vw_iv) + h$, and so $\beta \neq \square$. By Lemma 5, r is a multiple of $p'(vw_iv)$. Since $i \neq 1$, we have $(vw_iv)(h+r) = a$ by Lemma 8(1)(b) and so $a = (vw_iv)(h+r) = (vw_iv)(h + p'(vw_iv)) = \beta$. For Statement 4(c) with $\alpha = \square$, we get $((v'w')(h, \diamond)v')(h+r) = d$ since $d = (v'w'v')(h + p'(vw_iv)) = (v'w'v')(h+r)$, and so $q \in \mathcal{P}(u')$. For Statement 4(c) with $\alpha \neq \square$, we consider the case where $b = \alpha$ and then the case where $b \neq \alpha$. If $b = \alpha$, then $\alpha = b \neq a = \beta$ and $d = \overline{(v'w'v')(h - p'(vw_iv))}$, and so $((v'w')(h, \diamond)v')(h + r) = d$ since $d = (v'w'v')(h+p'(vw_iv)) = (v'w'v')(h+r)$. If $b \neq \alpha$, then $d = (v'w'v')(h - p'(vw_iv))$. In this case, $p'(vw_iv) = \gcd(p'(vw_iv), r) \in \mathcal{P}(vw_iv) = \mathcal{P}(v'w'v')$. We get $d = (v'w'v')(h - p'(vw_iv)) = (v'w'v')(h + p'(vw_iv)) = (v'w'v')(h + r)$ and so $((v'w')(h, \diamond)v')(h + r) = d$. For Statement 4(b), we get that r is a period of vw_1v and hence of $v'w'v'$ and so $((v'w')(h, \diamond)v')(h + r) = (v'w'v')(h + r) = (v'w'v')(h) = d$.

Case 5. $h + r > |vw_iv|$ and $r < h$ and $|vw_iv| \geq p'(vw_iv) + r$.

We have $|v| < r < h$ and so $x \neq \epsilon$ (otherwise $|vx| < r < h$ which is impossible since $h = |vx| + 1$). Here $\alpha \neq \square$ since $p'(vw_iv) \leq r < h$. By Lemma 5, r is a multiple of $p'(vw_iv)$. Since $i \neq k$, we have $(vw_iv)(h - r) = b$ by Lemma 8(1)(c) and so $b = (vw_iv)(h - r) = (vw_iv)(h - p'(vw_iv)) = \alpha$. For Statement 4(a), we get that r is a period of vw_kv and hence of $(v'w')(h, \bar{d})v' = v'w'v'$ and so $((v'w')(h, \diamond)v')(h - r) = \bar{d}$ since $(v'w'v')(h - r) = (v'w'v')(h) = \bar{d}$. For Statement 4(c), we get $((v'w')(h, \diamond)v')(h - r) = \bar{d}$ since $d = \overline{(v'w'v')(h - p'(vw_iv))}$.

Case 6. $h + r \leq |vw_iv|$ and $r < h$.

We have $|v| < r < h$ and so $x \neq \epsilon$ (otherwise $|vx| < r < h$ which is impossible since $h = |vx| + 1$). Here $|vw_iv| \geq p'(vw_iv) + r$ (otherwise, $r + p'(vw_iv) < h + p'(vw_iv) \leq h + r \leq |vw_iv| < p'(vw_iv) + r$, a contradiction). By Lemma 5, r is a multiple of $p'(vw_iv)$. We have $\alpha \neq \square$ since $p'(vw_iv) \leq r < h$, and $\beta \neq \square$ since $h + p'(vw_iv) \leq h + r \leq |vw_iv|$. Since $i \neq 1$, we have $(vw_iv)(h + r) = a$ by Lemma 8(1)(b) and so $a = (vw_iv)(h + r) =$

$(vw_iv)(h + p'(vw_iv)) = \beta$. Since $i \neq k$, we have $(vw_iv)(h - r) = b$ by Lemma 8(1)(c) and so $b = (vw_iv)(h - r) = (vw_iv)(h - p'(vw_iv)) = \alpha$. Then $\beta = a \neq b = \alpha$ and for Statement 4(c), we get $((v'w')(h, \diamond)v')(h + r) = d$ and $((v'w')(h, \diamond)v')(h - r) = \bar{d}$ since $d = \overline{(v'w'v')(h - p'(vw_iv))}$ and the result follows.

Case 7. $h + r \leq |vw_iv|$ and $r \geq h$ and $|vw_iv| < p'(vw_iv) + r$.

Here $\alpha = \square$ since $h + r \leq |vw_iv| < p'(vw_iv) + r$ and so $h < p'(vw_iv)$, but $\beta \neq \square$ since $h + p'(vw_iv) \leq h + r \leq |vw_iv|$. Since $i \neq 1$, we have $(vw_iv)(h + r) = a$ by Lemma 8(1)(b). For Statement 4(b), we get that r is a period of vw_1v and hence of $(v'w')(h, d)v' = v'w'v'$, and so $((v'w')(h, \diamond)v')(h + r) = (v'w'v')(h + r) = (v'w'v')(h) = d$. For Statement 4(c), we get that r is a period of vw_1v and hence of $(v'w')(h, d)v'$, and so $((v'w')(h, \diamond)v')(h + r) = ((v'w')(h, d)v')(h + r) = ((v'w')(h, d)v')(h) = d$.

Case 8. $h + r > |vw_iv|$ and $r < h$ and $|vw_iv| < p'(vw_iv) + r$.

Here $\beta = \square$ since $|vw_iv| < p'(vw_iv) + r < p'(vw_iv) + h$, but $\alpha \neq \square$ since $p'(vw_iv) \leq r < h$. Since $i \neq k$, we have $(vw_iv)(h - r) = b$ by Lemma 8(1)(c). For Statement 4(a), we get that r is a period of vw_kv and hence of $(v'w')(h, \bar{d})v' = v'w'v'$, and so $((v'w')(h, \diamond)v')(h - r) = (v'w'v')(h - r) = (v'w'v')(h) = \bar{d}$.

We show similarly that if $q \in \mathcal{P}(u')$ then $q \in \mathcal{P}(u)$. Note that for Statement 4(a), we have that $r \in \mathcal{P}'((v'w')(h, \diamond)v')$ and also $r \in \mathcal{P}((v'w')(h, \bar{d})v') = \mathcal{P}(v'w'v')$. It follows that $r \in \mathcal{P}(vw_kv)$ and $r \in \mathcal{P}'(vw_iv)$. For Statement 4(b), we have that $r \in \mathcal{P}'((v'w')(h, \diamond)v')$ and also $r \in \mathcal{P}((v'w')(h, d)v') = \mathcal{P}(v'w'v')$. It follows that $r \in \mathcal{P}(vw_1v)$ and $r \in \mathcal{P}'(vw_iv)$. For Statement 4(c) when $v'w'v'$ has a hole, we have that $r \in \mathcal{P}'((v'w')(h, \diamond)v') = \mathcal{P}'(v'w'v') = \mathcal{P}'(vw_iv)$. When $v'w'v'$ has no hole, we have that $\alpha = \square$. In this case, if $\beta = \square$, then $r \in \mathcal{P}'((v'w')(h, \diamond)v')$ and $r \in \mathcal{P}((v'w')(h, 0)v')$ and $r \in \mathcal{P}((v'w')(h, 1)v')$, and so $r \in \mathcal{P}(v'w'v')$ and $r \in \mathcal{P}'(vw_iv)$. If $\beta \neq \square$, then $r \in \mathcal{P}'((v'w')(h, \diamond)v')$ and $r \in \mathcal{P}((v'w')(h, d)v')$. So $r \in \mathcal{P}(v'w'v')$ by the definition of d. Hence $r \in \mathcal{P}'(vw_iv)$.

Last, let us show that $\mathcal{P}'(u') = \mathcal{P}'(u)$. Obviously, $|u| \in \mathcal{P}'(u)$ and $|u| \in \mathcal{P}'(u')$. Note that $p'(u) = |vw_1| = \cdots = |vw_k| = |v'w'|$ and so $p'(u) \in \mathcal{P}'(u)$ and $p'(u) \in \mathcal{P}'(u')$.

Consider q with $q \leq |u| - p'(u)$. If $q \in \mathcal{P}'(u)$, then Lemma 5 gives that q is a multiple of $p'(u)$, and therefore $q \in \{p'(u), 2p'(u), \ldots, (k - 1)p'(u)\}$. We get $q \in \mathcal{P}'(u')$ (for Statement 4, we get $q = p'(u)$). On the other hand, if $q \in \mathcal{P}'(u')$, then $|u'| = |u| \geq p'(u) + q$, and thus, by Theorem 1 or Theorem 2, $\gcd(p'(u), q) \in \mathcal{P}(u')$. Since $\mathcal{P}(u') = \mathcal{P}(u) \subseteq \mathcal{P}'(u)$, we get that $\gcd(p'(u), q) \in \mathcal{P}'(u)$. By the minimality of $p'(u)$, we have $\gcd(p'(u), q) = p'(u)$, and therefore $p'(u)$ divides q. We get $q \in \mathcal{P}'(u)$ (for Statement 4, we get $q = p'(u)$).

Now, consider q with $|u| - p'(u) < q < |u|$, and put $q = (k-1)p'(u) + r$ where $|v| < r < p'(u) + |v|$. We show that $\mathcal{P}'(u) \subseteq \mathcal{P}'(u')$ (the inclusion $\mathcal{P}'(u') \subseteq \mathcal{P}'(u)$ is proved similarly). If $q \in \mathcal{P}'(u)$, then $q \in \mathcal{P}(u)$. Since $\mathcal{P}(u) = \mathcal{P}(u')$, we get that $q \in \mathcal{P}(u')$ and hence $q \in \mathcal{P}'(u')$. □

Lemma 13 *Let u be as in Lemma 7(2) with $k > 1$.*

1. *Assume that $i = 1$. Put $Bin'(v_i w v_{i+1}) = v'' w' v'$ where $|v'| = |v''| = |v_i|$ and $|w'| = |w|$. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary partial word*

$$u' = v''(w'v')^k.$$

2. *Assume that $i = k+1$. Put $Bin'(v_{i-1} w v_i) = v' w' v''$ where $|v'| = |v''| = |v_i|$ and $|w'| = |w|$. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary partial word*

$$u' = (v'w')^k v''.$$

3. *Assume that $1 < i < k+1$ and $a = b$. Assume that $T(v_i) = [Bin'(v_i), \alpha, \beta]$ with $H(Bin'(v_i)) \subseteq H(v_i) = \{h\}$.*

   (a) *If $\alpha = \square$, then put $Bin'(v_{i-1} w v_i) = v' w' v''$ where $|v'| = |v''| = |v_i|$ and $|w'| = |w|$. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary partial word*

$$u' = (v'w')^{i-1} v'(h, \diamond)(w'v')^{k-i+1}.$$

   (b) *If $\alpha \neq \square$, then put $Bin'(v_i w v_{i+1}) = v'' w' v'$ where $|v'| = |v''| = |v_i|$ and $|w'| = |w|$. Then $\mathcal{P}'(u') = \mathcal{P}(u') = \mathcal{P}(u) = \mathcal{P}'(u)$, for the binary partial word*

$$u' = (v'w')^{i-1} v'(h, \diamond)(w'v')^{k-i+1}.$$

4. *Assume that $1 < i < k+1$ and $a \neq b$. Assume that $T(v_i) = [Bin'(v_i), \alpha, \beta]$ with $H(Bin'(v_i)) \subseteq H(v_i) = \{h\}$.*

   (a) *If ($\alpha = \square$ and $\beta \neq \square$ and $a = \beta$) or $x = \epsilon$, then put $Bin'(v_{i-1} w v_i) = v' w' v''$ where $|v'| = |v''| = |v_i|$ and $|w'| = |w|$, and put $d = v'(h)$. Then $\mathcal{P}(u') = \mathcal{P}(u)$ and $\mathcal{P}'(u') = \mathcal{P}'(u)$, for the binary partial word*

$$u' = (v'(h,d)w')^{i-1}v'(h,\diamond)(w'v'(h,\bar{d}))^{k-i+1}.$$

(b) If ($\alpha \neq \Box$ or $\beta = \Box$ or $a \neq \beta$) and $x \neq \epsilon$, then put $Bin'(v_i w v_{i+1}) = v''w'v'$ where $|v'| = |v''| = |v_i|$ and $|w'| = |w|$, and put $d = \overline{v'(h)}$. Then $\mathcal{P}(u') = \mathcal{P}(u)$ and $\mathcal{P}'(u') = \mathcal{P}'(u)$, for the binary partial word

$$u' = (v'(h,d)w')^{i-1}v'(h,\diamond)(w'v'(h,\bar{d}))^{k-i+1}.$$

Proof. For Statements 1, 2, and 3, the equality $\mathcal{P}'(u) = \mathcal{P}(u)$ is proved as in Lemma 12. For Statements 1, 2, 3, and 4, the equality $\mathcal{P}'(u') = \mathcal{P}'(u)$ follows as in Lemma 12 once the equality $\mathcal{P}(u') = \mathcal{P}(u)$ is proved.

First, let us show the equality $\mathcal{P}(u') = \mathcal{P}(u)$ for Statement 2 (Statement 1 is similar but uses Lemma 8(2)(a) instead of Lemma 8(2)(b)). The case where $q \in \mathcal{P}(u)$ with $q \leq |u| - p'(u)$ is proved as in Lemma 12. The case where $q \in \mathcal{P}(u')$ with $q \leq |u| - p'(u)$ is proved as follows. We have $|u'| = |u| \geq p'(u) + q$, and thus, by Theorem 1 or Theorem 2, $\gcd(p'(u), q) \in \mathcal{P}(u')$. Since $p'(u) = |v'w'|$ is a multiple of $\gcd(p'(u), q)$, we also have that $\gcd(p'(u), q)$ is a period of $v'w'v''$ and hence of $v_{i-1}wv_i$. So $\gcd(p'(u), q) \in \mathcal{P}(u)$ and since q is a multiple of $\gcd(p'(u), q)$, we also get $q \in \mathcal{P}(u)$.

The case where $|u| - p'(u) < q < |u|$ is proved as follows. Here $q = (k-1)p'(u) + r$ with $|v_i| < r < p'(u) + |v_i|$. By Lemma 8(2)(b), $q \in \mathcal{P}(u)$ if and only if $r \in \mathcal{P}'(v_{i-1}wv_i) = \mathcal{P}'(v'w'v'')$ which, by Lemma 4 or Lemma 8(2)(b), is equivalent with $q \in \mathcal{P}(u')$.

Now, let us show the equality $\mathcal{P}(u') = \mathcal{P}(u)$ for Statement 3. Again, the case where $q \in \mathcal{P}(u)$ with $q \leq |u| - p'(u)$ is proved as in Lemma 12. The case where $q \in \mathcal{P}(u')$ with $q \leq |u| - p'(u)$ is proved as follows. We have $|u'| = |u| \geq p'(u) + q$, and thus, by Theorem 2, $\gcd(p'(u), q) \in \mathcal{P}(u')$. For Statement 3(a), since $p'(u) = |v'w'|$ is a multiple of $\gcd(p'(u), q)$, we also have that $\gcd(p'(u), q)$ is a period of $v'w'v'$ and hence of $v'w'v''$ and of $v_{i-1}wv_i$. So $\gcd(p'(u), q) \in \mathcal{P}(u)$ and since q is a multiple of $\gcd(p'(u), q)$, we also get $q \in \mathcal{P}(u)$. The result follows similarly for Statement 3(b).

The case where $q \in \mathcal{P}(u)$ with $|u| - p'(u) < q < |u|$ is proved as follows (we show similarly the case where $q \in \mathcal{P}(u')$ with $|u| - p'(u) < q < |u|$). Here $q = (k-1)p'(u) + r$ with $|v_i| < r < p'(u) + |v_i|$. For Statement 3(a), the two conditions of Lemma 8(2)(b) hold. Here $r \in \mathcal{P}'(v_{i-1}wv_i)$ and hence $r \in \mathcal{P}'(v'w'v'')$ and $r \in \mathcal{P}'(v'w'v'(h,\diamond))$. If $r \geq h + p'(u)$, then $q \in \mathcal{P}(u')$ by Lemma 8(2)(b). If $r < h + p'(u)$, then $(v_{i-1}wv_i)(h + p'(u) - r) = b$ by

Lemma 8(2)(b)(ii). We consider the case where $r = p'(u)$, the case where $r > p'(u)$, and then the case where $r < p'(u)$. If $r = p'(u)$, then $(v'w'v'(h,\diamond))(h + p'(u) - p'(u)) = v'(h)$ and $q \in \mathcal{P}(u')$ by Lemma 8(2)(b). If $r > p'(u)$, then $r - p'(u)$ is a local period of v_i and so $r - p'(u) \geq p'(v_i)$. Since $r < h + p'(u)$, we get $r - p'(u) < h$. Since $\alpha = \square$, we get $r - p'(u) < h \leq p'(v_i)$ which yields a contradiction. If $r < p'(u)$, then $p'(u) - r$ is a local period of v_i and so $p'(v_i) \leq p'(u) - r$. If $h + p'(u) - r \leq |v_i|$, then $v_{i-1}(h) = a = b = (v_{i-1}wv_i)(h + p'(u) - r) = v_{i-1}(h + p'(u) - r)$ and so $v'(h) = v'(h + p'(u) - r)$. We conclude that $(v'w'v'(h,\diamond))(h + p'(u) - r) = v'(h + p'(u) - r) = v'(h)$, and $q \in \mathcal{P}(u')$ by Lemma 8(2)(b). If $h + p'(u) - r > |v_i|$, then $(v_{i-1}wv_i)(h + p'(u) - r) = b = a = v_{i-1}(h)$ and so $(v'w'v'(h,\diamond))(h + p'(u) - r) = v'(h)$ as desired.

For Statement 3(b), the two conditions of Lemma 8(2)(a) hold. Here $r \in \mathcal{P}'(v_iwv_{i+1})$ and hence $r \in \mathcal{P}'(v''w'v')$ and $r \in \mathcal{P}'(v'(h,\diamond)w'v')$. If $h + r > |v_iwv_{i+1}|$, then $q \in \mathcal{P}(u')$ by Lemma 8(2)(a). If $h + r \leq |v_iwv_{i+1}|$, then $(v_iwv_{i+1})(h + r) = a$ by Lemma 8(2)(a)(ii). We consider the case where $r = p'(u)$, the case where $r > p'(u)$, and then the case where $r < p'(u)$. If $r = p'(u)$, then $(v'(h,\diamond)w'v')(h+p'(u)) = v'(h)$ and $q \in \mathcal{P}(u')$ by Lemma 8(2)(a). If $r > p'(u)$, then $r-p'(u)$ is a local period of v_i and so $r-p'(u) \geq p'(v_i)$. Since $h+r \leq |v_iwv_{i+1}|$, we get $h + r - p'(u) \leq |v_i|$. We deduce that $h + p'(v_i) \leq h + r - p'(u) \leq |v_i|$ and so $\beta \neq \square$ and $v'' = v'(h,\diamond)$. We have $v_{i+1}(h) = a = (v_iwv_{i+1})(h + r) = (v_iwv_{i+1})(h + r - p'(u) + p'(u)) = v_{i+1}(h + r - p'(u))$ and so $v'(h) = v'(h + r - p'(u))$. We get $(v'(h,\diamond)w'v')(h + r) = (v'(h,\diamond)w'v')(h+r-p'(u)+p'(u)) = v'(h+r-p'(u)) = v'(h)$ and $q \in \mathcal{P}(u')$ by Lemma 8(2)(a). If $r < p'(u)$, then $p'(u)-r$ is a local period of v_i and so $p'(u)-r \geq p'(v_i)$. If $h+r-p'(u) > 0$, then $v_{i+1}(h) = a = (v_iwv_{i+1})(h+r) = (v_iwv_{i+1})(h+r-p'(u)+p'(u)) = v_{i+1}(h+r-p'(u)) = v_{i+1}(h - (p'(u) - r))$ and so $v'(h) = v'(h - (p'(u) - r))$. We get $(v'(h,\diamond)w'v')(h + r) = (v'(h,\diamond)w'v')(h + r - p'(u) + p'(u)) = v'(h + r - p'(u)) = v'(h - (p'(u) - r)) = v'(h)$, and $q \in \mathcal{P}(u')$ by Lemma 8(2)(a). If $h + r - p'(u) \leq 0$, then $(v_iwv_{i+1})(h + r) = a = b = v_{i+1}(h)$ and so $(v'(h,\diamond)w'v')(h + r) = v'(h)$ as desired.

Now, let us show the equality $\mathcal{P}(u') = \mathcal{P}(u)$ for Statement 4. Again, the case where $q \in \mathcal{P}(u)$ with $q \leq |u| - p'(u)$ is proved as in Lemma 12. The case where $q \in \mathcal{P}(u')$ with $q \leq |u| - p'(u)$ is proved as follows. We have $|u'| = |u| \geq p'(u) + q$, and thus, by Theorem 2, $\gcd(p'(u), q) \in \mathcal{P}(u')$. Since $p'(u) = |v'w'|$ is a multiple of $\gcd(p'(u), q)$, we also have that $\gcd(p'(u), q)$ is a period of $v'(h,d)w'v'(h,\bar{d})$ which is impossible.

The case where $q \in \mathcal{P}(u)$ with $|u|-p'(u) < q < |u|$ is proved as follows (we show similarly the case where $q \in \mathcal{P}(u')$ with $|u|-p'(u) < q < |u|$). Here $q = (k-1)p'(u)+r$ with $|v_i| < r < p'(u) + |v_i|$. For Statement 4(b), the two conditions of Lemma 8(2)(a) hold. In particular, $r \in \mathcal{P}'(v_iwv_{i+1})$ and hence $r \in \mathcal{P}'(v''w'v')$ and $r \in \mathcal{P}'(v'(h,\diamond)w'v'(h,\bar{d}))$. If $h+r > |v_iwv_{i+1}|$, then $q \in \mathcal{P}(u')$ by Lemma 8(2)(a). If $h + r \leq |v_iwv_{i+1}|$, then $r \neq p'(u)$ (otherwise, $a = b$ a contradiction). We consider the case where $r > p'(u)$ and then the case where $r < p'(u)$. If $r > p'(u)$, then $r-p'(u)$ is a local period of v_i and so $r-p'(u) \geq p'(v_i)$. Since $h+r \leq |v_iwv_{i+1}|$, we get $h+r-p'(u) \leq |v_i|$. We deduce that $h+p'(v_i) \leq h+r-p'(u) \leq |v_i|$ and so $\beta \neq \square$. We have $(v_iwv_{i+1})(h+r) = a$ by Lemma 8(2)(a)(ii), and so $v_{i+1}(h) = b \neq a = (v_iwv_{i+1})(h+r) = (v_iwv_{i+1})(h+r-p'(u)+p'(u)) = v_{i+1}(h+r-p'(u))$ and $v'(h) \neq v'(h+r-p'(u))$. We conclude that $(v'(h,\diamond)w'v'(h,\bar{d}))(h+r) = (v'(h,\diamond)w'v')(h+r) = (v'(h,\diamond)w'v')(h+r-p'(u)+p'(u)) = v'(h + r - p'(u)) = \overline{v'(h)} = d$, and $q \in \mathcal{P}(u')$ by Lemma 8(2)(a). If $r < p'(u)$, then $p'(u) - r$ is a local period of v_i and so $p'(u) - r \geq p'(v_i)$. If $h + r - p'(u) > 0$, then $v_{i+1}(h) = b \neq a = (v_iwv_{i+1})(h+r) = (v_iwv_{i+1})(h+r-p'(u)+p'(u)) = v_{i+1}(h+r-p'(u)) = v_{i+1}(h-(p'(u)-r))$ and so $v'(h) \neq v'(h-(p'(u)-r))$. We get $(v'(h,\diamond)w'v'(h,\bar{d}))(h+r) = (v'(h,\diamond)w'v')(h+r-p'(u)+p'(u)) = v'(h+r-p'(u)) = v'(h-(p'(u)-r)) = \overline{v'(h)} = d$, and $q \in \mathcal{P}(u')$ by Lemma 8(2)(a). If $h + r - p'(u) \leq 0$, then we argue as follows. If ($\alpha \neq \square$ or $\beta = \square$ or $a \neq \beta$) and $x \neq \epsilon$, then $(v_iwv_{i+1})(h + r) = a \neq b = v_{i+1}(h)$ and so $(v'(h,\diamond)w'v'(h,\bar{d}))(h+r) = \overline{v'(h,\bar{d})(h)} = d$ as desired.

For Statement 4(a), the two conditions of Lemma 8(2)(b) hold. In particular, $r \in \mathcal{P}'(v_{i-1}wv_i)$ and hence $r \in \mathcal{P}'(v'w'v'')$ and $r \in \mathcal{P}'(v'(h,d)w'v'(h,\diamond))$. If $r \geq h + p'(u)$, then $q \in \mathcal{P}(u')$ by Lemma 8(2)(b). If $r < h + p'(u)$, then $(v_{i-1}wv_i)(h + p'(u) - r) = b$ by Lemma 8(2)(b)(ii). Here $r \neq p'(u)$ (otherwise, $q = kp'(u)$ and $q \notin \mathcal{P}(u)$). If $r > p'(u)$, then the result follows as in Statement 3(a). If $r < p'(u)$, then $p'(u) - r$ is a local period of v_i and so $p'(v_i) \leq p'(u) - r$. If $h + p'(u) - r \leq |v_i|$, then $v_{i-1}(h) = a \neq b = (v_{i-1}wv_i)(h+p'(u)-r) = v_{i-1}(h+p'(u)-r)$ and so $v'(h) \neq v'(h+p'(u)-r)$. We conclude that $(v'(h,d)w'v'(h,\diamond))(h+p'(u)-r) = (v'w'v'(h,\diamond))(h+p'(u)-r) = v'(h+p'(u)-r) = \overline{v'(h)} = \bar{d}$, and $q \in \mathcal{P}(u')$ by Lemma 8(2)(b). If $h + p'(u) - r > |v_i|$, then we argue as follows. If $\alpha = \square$ and $\beta \neq \square$ and $a = \beta$, then $(v_{i-1}wv_i)(h + p'(u) - r) = b \neq a = \beta = v_i(h + p'(v_i))$ and so $(v_{i-1}wv_i)(h + p'(u) - r) \neq v_{i-1}(h)$ and $(v'(h,d)w'v'(h,\diamond))(h + p'(u) - r) = \overline{v'(h)} = \bar{d}$ as desired. If $x = \epsilon$, then $(v_{i-1}wv_i)(h+p'(u)-r) = b \neq a = v_{i-1}(h)$ and so $(v'(h,d)w'v'(h,\diamond))(h+$

$p'(u) - r) \neq v'(h)$. We get $(v'(h,d)w'v'(h,\diamond))(h + p'(u) - r) = \overline{v'(h)} = \bar{d}$ as desired. □

Theorem 4 *For every partial word u with one hole over an alphabet A, there exists a partial word v of length $|u|$ over the alphabet $\{0,1\}$ such that v does not begin with 1, $H(v) \subseteq H(u)$, $\mathcal{P}(v) = \mathcal{P}(u)$, and $\mathcal{P}'(v) = \mathcal{P}'(u)$.*

Proof. The proof is by induction on $|u|$. For $|u| \leq 3$, the result is obvious. Assume that the result holds for all partial words with one hole of length less than or equal to $n \geq 3$.

First, assume that u is as in Lemma 7(1) with $|u| = n + 1$. For $k = 1$, the word $\text{Bin}(v)$ satisfies $\mathcal{P}(\text{Bin}(v)) = \mathcal{P}(v)$. If $\text{Bin}(v) = \epsilon$, then $v = \epsilon$ and $u' = 01^{|u|-1}$ satisfies $\mathcal{P}(u') = \mathcal{P}(u)$ and $\mathcal{P}'(u') = \mathcal{P}'(u)$, since, in this case, $\mathcal{P}(u) = \mathcal{P}'(u) = \{|u|\}$. If $\text{Bin}(v) \neq \epsilon$, then $\text{Bin}(v)$ begins with 0. By Lemma 2, there exists $c \in \{0,1\}$ such that $\text{Bin}(v)1^{|w_1|-1}c$ is primitive. By Lemma 9, the word $u' = \text{Bin}(v)1^{|w_1|-1}c\text{Bin}(v)$ satisfies $\mathcal{P}(u') = \mathcal{P}(u)$ and $\mathcal{P}'(u') = \mathcal{P}'(u)$. For $k > 1$, the result follows by Lemma 12. We have $|vw_iv| \leq n$ and, by the inductive hypothesis, there exists a partial word $\text{Bin}'(vw_iv)$ over the alphabet $\{0,1\}$ such that $\text{Bin}'(vw_iv)$ begins with 0 or $\diamond$, $H(\text{Bin}'(vw_iv)) \subseteq H(vw_iv) = \{h\}$, $\mathcal{P}(\text{Bin}'(vw_iv)) = \mathcal{P}(vw_iv)$, and $\mathcal{P}'(\text{Bin}'(vw_iv)) = \mathcal{P}'(vw_iv)$. Consider for instance the case where $1 < i < k$ and $a \neq b$ and $\beta_{vw_iv} \neq \square$ and $x \neq \epsilon$. By the inductive hypothesis, there exist v' and w' over the alphabet $\{0,1\}$ such that $\text{Bin}'(vw_iv) = v'w'v'$, $|v'| = |v|$ and $|w'| = |w_i|$. The partial word

$$u' = ((v'w')(h,d))^{i-1}(v'w')(h,\diamond)((v'w')(h,\bar{d}))^{k-i}v'$$

where d is defined as in Lemma 12(4)(c) satisfies the desired properties. In particular, u' begins with 0. To see this, since $x \neq \epsilon$ we have $h > 1$, and since $v'w'v'$ begins with 0 the result follows. The other cases are handled similarly.

Now, assume that u is as in Lemma 7(2) with $|u| = n+1$. For $k = 1$, first say $v_1 = x\diamond y$ and $v_2 = xby$ (here $u = x\diamond ywxby$). We have $|v_1| \leq n$ and, by the inductive hypothesis, there exists a partial word $\text{Bin}'(v_1)$ over the alphabet $\{0,1\}$ such that $\text{Bin}'(v_1)$ begins with 0 or $\diamond$, $H(\text{Bin}'(v_1)) \subseteq H(v_1)$, $\mathcal{P}(\text{Bin}'(v_1)) = \mathcal{P}(v_1)$, and $\mathcal{P}'(\text{Bin}'(v_1)) = \mathcal{P}'(v_1)$. If $\beta_{v_1} \neq \square$, then Lemma 10(2) shows the existence of binary numbers c and d such that the partial word $u' = \text{Bin}'(v_1)1^{|w|-1}c\text{Bin}'(v_1)(H(\text{Bin}'(v_1)), d)$ satisfies the desired properties. If $\beta_{v_1} = \square$, then the result follows by Lemma 10(1).

Now say $v_1 = xay$ and $v_2 = x\diamond y$ (here $u = xaywx\diamond y$). We have $|v_2| \leq n$ and, by the inductive hypothesis, there exists a partial word $\text{Bin}'(v_2)$ over the alphabet $\{0,1\}$

such that $\mathrm{Bin}'(v_2)$ does not begin with 1, $H(\mathrm{Bin}'(v_2)) \subseteq H(v_2)$, $\mathcal{P}(\mathrm{Bin}'(v_2)) = \mathcal{P}(v_2)$, and $\mathcal{P}'(\mathrm{Bin}'(v_2)) = \mathcal{P}'(v_2)$. We first consider the case where $\alpha_{v_2} \neq \Box$. In this case, $H(\mathrm{Bin}'(v_2)) \neq \{1\}$. For d defined as in Lemma 11(2), by Lemma 2, there exists $c \in \{0,1\}$ such that $\mathrm{Bin}'(v_2)(H(\mathrm{Bin}'(v_2)), d)1^{|w|-1}c$ is primitive (put $c = 1$ if $\mathrm{Bin}'(v_2) = 0^{|x|} \diamond 1^{|y|}$ in which case $x \neq \epsilon$). By Lemma 11(2), the partial word $u' = \mathrm{Bin}'(v_2)(H(\mathrm{Bin}'(v_2)), d)1^{|w|-1}c\mathrm{Bin}'(v_2)$ satisfies the desired properties. The case where $\alpha_{v_2} = \Box$ follows from Lemma 11(1).

For $k > 1$, the result follows by Lemma 13. For the case where $1 < i < k+1$ and $a \neq b$ for instance, by Lemma 13(4), we have $|v_i| \leq n$ and, by the inductive hypothesis, there exists a partial word $\mathrm{Bin}'(v_i)$ over the alphabet $\{0,1\}$ such that $\mathrm{Bin}'(v_i)$ begins with 0 or $\diamond$, $H(\mathrm{Bin}'(v_i)) \subseteq H(v_i) = \{h\}$, $\mathcal{P}(\mathrm{Bin}'(v_i)) = \mathcal{P}(v_i)$, and $\mathcal{P}'(\mathrm{Bin}'(v_i)) = \mathcal{P}'(v_i)$. Consider for instance the case where ($\alpha_{v_i} \neq \Box$ or $\beta_{v_i} = \Box$ or $a \neq \beta_{v_i}$) and $x \neq \epsilon$. Then by Lemma 13(4)(b), since $|v_i w v_{i+1}| \leq n$, by the inductive hypothesis, there exist v', w', and v'' over the alphabet $\{0,1\}$ such that $\mathrm{Bin}'(v_i w v_{i+1}) = v''w'v'$, $|v'| = |v''| = |v_i|$ and $|w'| = |w|$, $v''w'v'$ begins with 0, $H(v''w'v') \subseteq H(v_i w v_{i+1}) = \{h\}$, $\mathcal{P}(v''w'v') = \mathcal{P}(v_i w v_{i+1})$, and $\mathcal{P}'(v''w'v') = \mathcal{P}'(v_i w v_{i+1})$. The partial word

$$u' = (v'(h,d)w')^{i-1}v'(h,\diamond)(w'v'(h,\bar{d}))^{k-i+1}$$

where $d = \overline{v'(h)}$ satisfies the desired properties. In particular, u' begins with 0 since $h \neq 1$ and v' begins with 0. $\Box$

6 Our algorithm

We now describe our algorithm.

Algorithm 2 *Let A be an alphabet not containing the special symbol $\Box$. Given as input a partial word u with one hole over A, put $H(u) = \{h\}$ where $1 \leq h \leq |u|$. The following algorithm computes a triple $T(u) = [Bin'(u), \alpha_u, \beta_u]$, where $Bin'(u)$ is a partial word of length $|u|$ over the alphabet $\{0,1\}$ such that $Bin'(u)$ does not begin with 1, where $H(Bin'(u)) \subseteq \{h\}$, where $\mathcal{P}(Bin'(u)) = \mathcal{P}(u)$ and $\mathcal{P}'(Bin'(u)) = \mathcal{P}'(u)$, and where*

$$\alpha_u = \begin{cases} \Box & \text{if } h - p'(u) < 1, \\ u(h - p'(u)) & \text{otherwise,} \end{cases}$$

and

$$\beta_u = \begin{cases} \Box & \text{if } h + p'(u) > |u|, \\ u(h + p'(u)) & \text{otherwise.} \end{cases}$$

Moreover, if $\mathcal{P}(u) \neq \mathcal{P}'(u)$, then $H(Bin'(u)) = \{h\}$ and $\alpha_u = u(h - p'(u)) \neq u(h + p'(u)) = \beta_u$. Also, if $\alpha_u \neq \Box$ and $\beta_u \neq \Box$, then $H(Bin'(u)) = \{h\}$.

Find the minimal local period $p'(u)$ of u. If $p'(u) = |u|$, then output $T(u) = [01^{|u|-1}, \Box, \Box]$. If $p'(u) \neq |u|$, then find partial words satisfying Lemma 7(1) or Lemma 7(2).

1. *If the partial words found satisfy Lemma 7(1), then do one of the following:*

   (a) *If $k = 1$, then compute $Bin(v)$, find $c \in \{0, 1\}$ such that $Bin(v)1^{|w_1|-1}c$ is primitive, and output*

   $$T(u) = [Bin(v)1^{|w_1|-1}cBin(v), \Box, \Box].$$

   (b) *If $k > 1$, then compute $T(vw_iv) = [Bin'(vw_iv), \alpha, \beta]$ and compute $h' = h - (i-1)p'(u)$. Then do one of the following:*

      i. *If $i = 1$, then do one of the following:*

         A. *If $\beta = \Box$, then compute $Bin(vw_kv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_k|$. Then output*

         $$T(u) = [(v'w')^k v', \Box, b].$$

         B. *If $\alpha = \Box$ and $\beta \neq \Box$, then compute $Bin'(vw_iv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_i|$. Find $d \in \{0, 1\}$ as follows:*

         $$d = \begin{cases} 1 & \text{if } v = \epsilon \text{ and } x \neq \epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

         and output

         $$T(u) = [(v'w')(h', \diamond)((v'w')(h', \bar{d}))^{k-1}v', \Box, b].$$

         C. *If $\alpha \neq \Box$ and $\beta \neq \Box$, then compute $Bin'(vw_iv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_i|$. Find $d \in \{0, 1\}$ as follows:*

         $$d = \begin{cases} \overline{Bin'(vw_iv)(h' - p'(vw_iv))} & \text{if } b = \alpha, \\ Bin'(vw_iv)(h' - p'(vw_iv)) & \text{if } b \neq \alpha, \end{cases}$$

         and output

$$T(u) = [v'w'((v'w')(h', \bar{d}))^{k-1}v', \Box, b].$$

ii. *If $i = k$, then do one of the following:*

A. *If $\alpha = \Box$, then compute Bin$(vw_1v) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_1|$. Then output*

$$T(u) = [(v'w')^k v', a, \Box].$$

B. *If $\alpha \neq \Box$ and $\beta = \Box$, then compute Bin$'(vw_iv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_i|$. Find $d \in \{0, 1\}$ as follows:*

$$d = \begin{cases} 1 & \textit{if } v = \epsilon \textit{ and } y \neq \epsilon, \\ 0 & \textit{otherwise,} \end{cases}$$

and output

$$T(u) = [((v'w')(h', d))^{k-1}(v'w')(h', \diamond)v', a, \Box].$$

C. *If $\alpha \neq \Box$ and $\beta \neq \Box$, then compute Bin$'(vw_iv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_i|$. Find $d \in \{0, 1\}$ as follows:*

$$d = \begin{cases} \mathit{Bin}'(vw_iv)(h' + p'(vw_iv)) & \textit{if } a = \beta, \\ \overline{\mathit{Bin}'(vw_iv)(h' + p'(vw_iv))} & \textit{if } a \neq \beta, \end{cases}$$

and output

$$T(u) = [((v'w')(h', d))^{k-1}v'w'v', a, \Box].$$

iii. *If $1 < i < k$ and $a = b$, then compute Bin$(vw_1v) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_1|$. Then output*

$$T(u) = [(v'w')^{i-1}(v'w')(h', \diamond)(v'w')^{k-i}v', a, b].$$

iv. *If $1 < i < k$ and $a \neq b$, then do one of the following:*

A. *If $\alpha \neq \Box$ and $\beta = \Box$, then compute Bin$(vw_kv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_k|$, and put $d = \overline{(v'w')(h')}$. Then output*

$$T(u) = [((v'w')(h', d))^{i-1}(v'w')(h', \diamond)((v'w')(h', \bar{d}))^{k-i}v', a, b].$$

B. *If $\beta \neq \Box$ and $x = \epsilon$, then compute Bin$(vw_1v) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_1|$, and put $d = (v'w')(h')$. Then output*

$$T(u) = [((v'w')(h', d))^{i-1}(v'w')(h', \diamond)((v'w')(h', \bar{d}))^{k-i}v', a, b].$$

C. *If ($\alpha = \Box$ and $\beta = \Box$) or ($\beta \neq \Box$ and $x \neq \epsilon$), then compute Bin$'(vw_iv) = v'w'v'$ where $|v'| = |v|$ and $|w'| = |w_i|$. Find $d \in \{0, 1\}$ as follows:*

$$d = \begin{cases} \overline{Bin'(vw_iv)(h' - p'(vw_iv))} & \text{if } \alpha \neq \square \text{ and } b = \alpha \text{ and } a = \beta, \\ Bin'(vw_iv)(h' - p'(vw_iv)) & \text{if } \alpha \neq \square \text{ and } (b \neq \alpha \text{ or } a \neq \beta), \\ Bin'(vw_iv)(h' + p'(vw_iv)) & \text{if } \alpha = \square \text{ and } \beta \neq \square \text{ and } a = \beta, \\ \overline{Bin'(vw_iv)(h' + p'(vw_iv))} & \text{if } \alpha = \square \text{ and } \beta \neq \square \text{ and } a \neq \beta, \\ 0 & \text{otherwise,} \end{cases}$$

and output

$$T(u) = [((v'w')(h', d))^{i-1}(v'w')(h', \diamond)((v'w')(h', \bar{d}))^{k-i}v', a, b].$$

2. *If the partial words found satisfy Lemma 7(2), then do one of the following:*

   (a) *If $k = 1$, then do one of the following:*

      i. *If $v_1 = x\diamond y$ and $v_2 = xby$, compute $T(v_1) = [Bin'(v_1), \alpha, \beta]$.*

         A. *If $\beta = \square$, then compute $Bin(v_2)$, find $c \in \{0,1\}$ such that $Bin(v_2)1^{|w|-1}c$ is primitive, and output*

         $$T(u) = [Bin(v_2)1^{|w|-1}cBin(v_2), \square, b].$$

         B. *If $\beta \neq \square$, then find $d \in \{0,1\}$ as follows:*

         $$d = \begin{cases} \overline{Bin'(v_1)(h - p'(v_1))} & \text{if } \alpha \neq \square \text{ and } b = \alpha, \\ Bin'(v_1)(h - p'(v_1)) & \text{if } \alpha \neq \square \text{ and } b \neq \alpha, \\ 1 & \text{otherwise.} \end{cases}$$

         Find $c \in \{0,1\}$ as follows. If $Bin'(v_1) = 0^{|x|}\diamond 1^{|y|}$, let $c = 1$. Otherwise, if $Bin'(v_1)1^{|w|-1}$ is not of the form $z\diamond z$, then let c be such that $Bin'(v_1)1^{|w|-1}c$ is primitive. Otherwise, let $c = \bar{d}$. Then output

         $$T(u) = [Bin'(v_1)1^{|w|-1}cBin'(v_1)(H(Bin'(v_1)), d), \square, b].$$

      ii. *If $v_1 = xay$ and $v_2 = x\diamond y$, compute $T(v_2) = [Bin'(v_2), \alpha, \beta]$. Compute $h' = h - p'(u)$.*

         A. *If $\alpha = \square$, then compute $Bin(v_1)$, find $c \in \{0,1\}$ such that $Bin(v_1)1^{|w|-1}c$ is primitive, and output*

         $$T(u) = [Bin(v_1)1^{|w|-1}cBin(v_1), a, \square].$$

         B. *If $\alpha \neq \square$, then find $d \in \{0,1\}$ as follows:*

         $$d = \begin{cases} Bin'(v_2)(h' + p'(v_2)) & \text{if } \beta \neq \square \text{ and } a = \beta, \\ \overline{Bin'(v_2)(h' + p'(v_2))} & \text{if } \beta \neq \square \text{ and } a \neq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Find $c \in \{0,1\}$ as follows. If $Bin'(v_2) = 0^{|x|}\diamond 1^{|y|}$, let $c = 1$. Otherwise, let c be such that $Bin'(v_2)(H(Bin'(v_2)), d)1^{|w|-1}c$ is primitive. Then output

$$T(u) = [Bin'(v_2)(H(Bin'(v_2)), d)1^{|w|-1}cBin'(v_2), a, \Box].$$

(b) *If $k > 1$, then do one of the following:*

i. *If $i = 1$, compute $Bin'(v_i w v_{i+1}) = v''w'v'$ where $|v'| = |v''| = |v_i|$ and $|w'| = |w|$, and output*

$$T(u) = [v''(w'v')^k, \Box, b].$$

ii. *If $i = k+1$, compute $Bin'(v_{i-1} w v_i) = v'w'v''$ where $|v'| = |v''| = |v_i|$ and $|w'| = |w|$, and output*

$$T(u) = [(v'w')^k v'', a, \Box].$$

iii. *If $1 < i < k+1$ and $a = b$, compute $T(v_i) = [Bin'(v_i), \alpha, \beta]$ and compute $h' = h - (i-1)p'(u)$. Then do one of the following:*

A. *If $\alpha = \Box$, compute $Bin'(v_{i-1} w v_i) = v'w'v''$ where $|v'| = |v''| = |v_i|$ and $|w'| = |w|$, and output*

$$T(u) = [(v'w')^{i-1}v'(h', \diamond)(w'v')^{k-i+1}, a, b].$$

B. *If $\alpha \neq \Box$, compute $Bin'(v_i w v_{i+1}) = v''w'v'$ where $|v'| = |v''| = |v_i|$ and $|w'| = |w|$, and output*

$$T(u) = [(v'w')^{i-1}v'(h', \diamond)(w'v')^{k-i+1}, a, b].$$

iv. *If $1 < i < k+1$ and $a \neq b$, compute $T(v_i) = [Bin'(v_i), \alpha, \beta]$, and compute $h' = h - (i-1)p'(u)$. Then do one of the following:*

A. *If ($\alpha = \Box$ and $\beta \neq \Box$ and $a = \beta$) or $x = \epsilon$, compute $Bin'(v_{i-1} w v_i) = v'w'v''$ where $|v'| = |v''| = |v_i|$ and $|w'| = |w|$, and put $d = v'(h')$. Then output*

$$T(u) = (v'(h', d)w')^{i-1}v'(h', \diamond)(w'v'(h', \bar{d}))^{k-i+1}, a, b].$$

B. *If ($\alpha \neq \Box$ or $\beta = \Box$ or $a \neq \beta$) and $x \neq \epsilon$, compute $Bin'(v_i w v_{i+1}) = v''w'v'$ where $|v'| = |v''| = |v_i|$ and $|w'| = |w|$, and put $d = \overline{v'(h')}$. Then output*

$$T(u) = [(v'(h', d)w')^{i-1}v'(h', \diamond)(w'v'(h', \bar{d}))^{k-i+1}, a, b].$$

We now give a few examples.

Example 2 *1. If $u = abb\diamond cbb$, then $T(u) = [0111111, \Box, \Box]$. Both u and $Bin'(u)$ have only the period 7 and the local period 7.*

2. *If* $u = (abb)(c\diamond ca)(abb) = vw_1v$, *then* $T(u) = [0111111011, \square, \square]$. *Both* u *and* $Bin'(u)$ *have only the periods 7, 10 and the local periods 7, 10. This example illustrates item 1(a).*

3. *If* $u = (abcdabfdabfd)(abcd)(abcdab\diamond dabfd)(abcd)(abcdabcdabfd) = v_1wv_2wv_3$, *then*

$$T(v_2) = [010101\diamond 10111, c, f]$$

and $T(u) = [010101110111111010101\diamond 1011111110101010111, f, c]$. *Both* u *and* $Bin'(u)$ *have only the periods 36, 44 and the local periods 16, 36, 44. This example illustrates Item 2(b)(iv)(B).*

4. *If* $u = (abcdabedabcd)(ab\diamond dabedabcd)(abedabedabcd) = w_1w_2w_3$, *then*

$$T(w_2) = [011111110111, \square, c]$$

and $T(u) = [01111111011101\diamond 11111011101011110111, c, e]$. *Both* u *and* $Bin'(u)$ *have only the periods 32, 36 and the local periods 12, 32, 36. This example illustrates Item 1(b)(iv)(C).*

5. *If* $u = (adabc)(d)(\diamond dabc)(d)(adabc) = v_1wv_2wv_3$, *then*

$$T(v_1wv_2) = [01111101111, a, \square]$$

and $T(u) = [011111\diamond 1111101111, a, a]$. *Both* u *and* $Bin'(u)$ *have only the periods 6, 12, 17 and the local periods 6, 12, 17. This example illustrates Item 2(b)(iii)(A).*

The correcteness of our algorithm follows from the proof of Theorem 4. We now consider the complexity of our algorithm.

Theorem 5 *Algorithm 2 runs in linear time and therefore is optimal.*

Proof. Let us first compute the complexity of the main functions of Algorithm 2.

- *Compute the minimal local period:* Let us consider finding the minimal local period of a partial word with one hole. Halava, Harju and Ilie [10] showed how a linear pattern matching algorithm can be easily adapted to compute the minimal period of a given word u. Given words v and w, their algorithm finds the leftmost occurrence, if any, of v as a factor of w. The comparisons done by their algorithm are of the type $a \stackrel{?}{=} b$, for letters a and b. Their algorithm can be easily adapted to compute $p'(u)$ for a partial word u with one hole by overloading the comparison operator in $a \stackrel{?}{=} b$ to return all comparisons of the special symbol $\diamond$ with any letter a or b as true. (For example, both $\diamond \stackrel{?}{=} b$ and $a \stackrel{?}{=} \diamond$ returns true for all letters a and b in the alphabet A, while $a \stackrel{?}{=} b$ only returns true if both a and b are the same symbol.) Overloading the operator does not change the time complexity of the algorithm any more than by a constant factor. Thus, the computing of $p'(u)$ can be performed in linear time.

- *Find partial words satisfying Lemma 7:* Finding a positive integer k and partial words $v, w_1, w_2, \ldots, w_k$ satisfying Lemma 7(1) (respectively, finding a positive integer k and partial words $w, v_1, v_2, \ldots, v_{k+1}$ satisfying Lemma 7(2)) is performed in linear time, since we know that $p'(u) = |vw_1| = |vw_2| = \cdots = |vw_k|$ (respectively, $p'(u) = |v_1w| = |v_2w| = \cdots = |v_kw| = |v_{k+1}w|$) from computing the minimal local period as described above.

- *Test for primitivity:* It is well known that primitivity can be tested in linear time for binary full words [6]. Indeed, a word u is primitive if and only if $u^2 = xuy$ implies that either $x = \epsilon$ or $y = \epsilon$. This part of the algorithm needs to be altered slightly to handle binary partial words with one hole. By far the easiest approach would be to substitute the hole with a 0 and test the new binary full word for primitivity as above. If the new word is primitive, then substitute the hole for 1 and test this new word for primitivity. If both words are primitive, then the binary partial word with one hole is primitive, otherwise it is not. This change in the algorithm increases the time complexity by at most a constant factor.

Algorithm 2 also uses the linear algorithm by Halava, Harju and Ilie (Algorithm 1) for constructing binary images of given words via Bin.

Algorithm 2 is recursive, so let us compute the complexity of a single call of the procedure T, say $f(n)$, where n is the length of the current partial word for this call, say u. Let us consider the call related to Item 2(a)(i)(A) (the other items are handled similarly). There, u satisfies Lemma 7(2) with $k = 1$, $v_1 = x\diamond y$ and $v_2 = xby$. Algorithm 2 computes the following functions on u:

1. Compute $p'(u)$.

2. Find partial words satisfying Lemma 7.

3. Compute $\text{Bin}(v_2)$.

4. Test for primitivity.

Since each function of the worst case of Algorithm 2 is linear, we have shown so far that a single call of T requires $f(n) = O(n)$ time. (Every function used in our algorithm requires at most linear time.) More precisely, there is a constant c such that $f(n) \leq cn$, for any $n \geq 0$.

To calculate the time required for the whole algorithm on an input u of length n, we first determine how fast the length of the current partial word decreases from a call to the next call or the next two calls. Let us examine the worst case of Lemma 7(2) following path 2(b)(iii)(X) or 2(b)(iv)(X) with X being any subcase. Consider u_1 and u_2 the current partial words for two consecutive calls of T on u, respectively. For instance, for 2(b)(iv)(A), we have that $u = v_1wv_2w \ldots v_kwv_{k+1}$, $u_1 = v_i$, and $u_2 = v_{i-1}wv_i$, and consequently $|u_1| < |u_2| \leq 2/3|u|$. Therefore, the time required by Algorithm 2 to compute $\text{Bin}'(u)$ is at most

$$2\Sigma_{i\geq 0}f((2/3)^in) \leq 2\Sigma_{i\geq 0}c(2/3)^in \leq 6cn,$$

hence it is linear, as claimed. Finally, it is clear that our algorithm is optimal, as the problem requires at least linear time. □

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