Recurrence in Infinite Partial Words*

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Abstract

The recurrence function $R_w(n)$ of an infinite word $w$ was introduced by Morse and Hedlund in relation to symbolic dynamics. It is the size of the smallest window such that, wherever its position on $w$, all length $n$ subwords of $w$ will appear at least once inside that window. The recurrence quotient $\rho(w)$ of $w$, defined as $\limsup \frac{R_w(n)}{n}$, is useful for studying the growth rate of $R_w(n)$. It is known that if $w$ is periodic, then $\rho(w) = 1$, while if $w$ is not, then $\rho(w) \geq 3$. A long standing conjecture from Rauzy states that the latter can be improved to $\rho(w) \geq \frac{3+\sqrt{5}}{2} \sim 3.618$, this bound being true for each Sturmian word and being reached by the Fibonacci word. In this paper, we study in particular the spectrum of values taken by the recurrence quotients of infinite partial words, which are sequences that may have some undefined positions. In this case, we determine exactly the spectrum of values, which turns out to be 1, every real number greater than or equal to 2, and $\infty$. More precisely, if an infinite partial word $w$ is “ultimately factor periodic”, then $\rho(w) = 1$, while if $w$ is not, then $\rho(w) \geq 2$, and we give constructions of infinite partial words achieving each value.

Keywords: Automata and Formal Languages; Combinatorics on Words; Partial Words; Recurrent Words; Recurrence Quotient; Recurrence Function.

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1 Introduction

A topic of interest on infinite words is the one of recurrence. An infinite recurrent word is one in which all of the finite subwords appear infinitely often, where finite subwords are finite contiguous blocks of letters. Many concepts dealing with recurrence were introduced by Morse and Hedlund in [8] in relation to symbolic dynamics. More recently, in [5], Cassaigne presented some results that establish connections between recurrence and the subword complexity $p_w(n)$ of an infinite word $w$, which is the number of distinct subwords of length $n$ in $w$. He also described, under some conditions, a method for computing the recurrence function $R_w(n)$ of an infinite word $w$, which is the minimum length such that every contiguous block of letters in $w$ of this length contains every length $n$ subword of $w$.

The recurrence quotient of an infinite word $w$ is defined to be $\rho(w) = \limsup_n \frac{R_w(n)}{n}$. Cassaigne studied in [4] the spectrum of possible recurrence quotients for Sturmian words (those with subword complexity $n + 1$ [7]). It is a compact subset of $[0, \infty]$ with empty interior, it has cardinality of the continuum, and its smallest accumulation point is approximately $4.58565$. He discussed in [5] the spectrum of values $S \subset \mathbb{R} \cup \{\infty\}$ taken by $\rho$ for arbitrary words, about which much less is known. Periodic words, those of the form $x\omega$, have a quotient $\rho$ of 1, and he proved, using graph representations, that $\rho \geq 3$ otherwise.

**Theorem 1** ([5]). Lower bounds for the recurrence quotients achievable by infinite words $w$ are:

- If $w$ is periodic, then $\rho(w) = 1$;
- If $w$ is nonperiodic, then $\rho(w) \geq 3$.

Note that this bound of 3 improved an earlier bound of 2 that can be easily deduced from Hedlund and Morse’s inequalities that $R_w(n) \geq p_w(n) + n − 1$ and $p_w(n) \geq n + 1$ for nonperiodic recurrent words [8] (as was mentioned in [5], if $w$ is a recurrent infinite word that is not periodic, then $R_w(n) \geq 2n$). However, the bound of 3 is not tight and Rauzy conjectured that the minimum value for nonperiodic words is $\frac{5+\sqrt{5}}{2} \approx 3.618$, which is achieved by the well-known Fibonacci word

$$01001001001010010101001001001001\cdots$$

defined by $F_{n+2} = F_{n+1}F_n$, where $F_0 = 0$ and $F_1 = 1$ [10]. Very little else is known about the topological structure of $S$. 

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How do these results related to recurrence in infinite words translate to the framework of infinite partial words? Partial words are sequences over a finite alphabet that may contain wildcard symbols, called holes, which match or are compatible with all letters; partial words without holes are said to be full words (or simply words). Combinatorics on partial words was initiated by Berstel and Boasson in the context of gene comparison [1] and has been developing since (see for instance [2]). In [3], Blanchet-Sadri et al. introduced recurrent infinite partial words. Their results culminate in showing that the various completions of an infinite partial word \( w \), i.e., infinite full words built by filling in the holes of \( w \) with letters from the alphabet, can achieve subword complexities equal or close to that of \( w \) if and only if \( w \) is recurrent or ultimately recurrent (ultimate recurrence of \( w \) means recurrence of some suffix of \( w \)).

As mentioned earlier, the exact spectrum \( S \) of achievable recurrence quotients for infinite full words is not known, although it is known to be included in \( \{1\} \cup [3, \infty] \) (see Theorem 1). In this paper, we consider the recurrence quotients of infinite partial words. Our main result, Theorem 2, states that the exact spectrum \( S_\diamond \) of achievable recurrence quotients for infinite partial words is \( \{1\} \cup [2, \infty] \).

**Theorem 2.** The spectrum of recurrence quotients achievable by partial words is \( S_\diamond = \{1\} \cup [2, \infty] \). More precisely, the following hold for infinite partial words \( w \):

- If \( w \) is ultimately factor periodic, then \( \rho(w) = 1 \);
- If \( w \) is not ultimately factor periodic, then \( \rho(w) \in [2, \infty] \).

Note that to obtain different spectra in the context of partial words, we need to consider factor periodicity and not just periodicity as in the context of full words (see Section 2 for the definition of these terms). Moreover, we give constructions of infinite partial words achieving each value in \( S_\diamond \). We also provide some results showing how the distribution of holes in an recurrent infinite partial word \( w \) implies the nonultimate periodicity of \( w \), strenghtening some results in [3].

The contents of our paper is as follows: In Section 2, we introduce our notations and terminology on partial words. In Section 3, we study the spectrum of values for recurrence quotients of infinite partial words. In Section 3.1 we prove that only the values in \( \{1\} \cup [2, \infty] \) are possible recurrence quotients, and in Section 3.2 we give explicit constructions achieving each value. In Section 4, we give some properties of recurrent partial words.
We present some relations between recurrence and periodicity as well as a relation between uniform recurrence and subword complexity. Finally in Section 5, we conclude with some remarks.

2 Preliminaries

For more information on basics of partial words, we refer the reader to [2]. Unless explicitly stated, $A$ is a finite alphabet that contains at least two distinct letters 0 and 1. We denote the set of all words over $A$ by $A^*$, which under the concatenation operation forms a free monoid whose identity is the empty word $\varepsilon$.

A finite partial word of length $n$ over $A$ is a function $w : \{0, \ldots, n-1\} \rightarrow A \cup \{\diamond\}$, where $\diamond \not\in A$. The union set $A \cup \{\diamond\}$ is denoted by $A_\diamond$. The length of $w$ is denoted by $|w|$. A right infinite partial word or infinite partial word over $A$ is a function $w : \mathbb{N} \rightarrow A_\diamond$. In both the finite and infinite cases, the symbol at position $i$ in $w$ is denoted by $w_i$. If $w_i \in A$, then $i$ is defined in $w$, and if $w_i = \diamond$, then $i$ is a hole in $w$. If $w$ has no holes, then $w$ is a full word. Two finite partial words $u$ and $v$ of same length are compatible, denoted $u \uparrow v$, if $u_i = v_i$ whenever $u_i, v_i \in A$. Equivalently, $u$ and $v$ are compatible if there exists a full word $w$ which is a completion of both $u$ and $v$, that is, we can fill the holes of $u$ and $v$ and obtain $w$ in either case.

Let $w = w_0w_1w_2 \cdots$ be an infinite partial word over $A$. We say that the finite partial word $u$ is a factor of $w$ if $u$ is a block of consecutive symbols of $w$, that is, $w = xuy$ for some partial words $x, y$. We say that the finite full word $v$ is a subword of $w$ if $v$ is compatible with some factor of $w$. We write $\text{Sub}_w(n)$ to denote the set of length $n$ subwords of $w$, and $\text{Sub}(w)$ to denote the set of all subwords of $w$. The subword complexity of $w$ is the function defined by $p_w(n) = |\text{Sub}_w(n)|$. For example, if $w = 001000\diamond010\diamond1\cdots$ and $A = \{0, 1\}$, then 00001 and 00101 are the subwords compatible with the underlined factor of $w$. We can check that $\text{Sub}_w(2) = \{00, 01, 10, 11\}$ and $p_w(2) = 4$.

We call an infinite partial word $w$ recurrent if every subword appears infinitely often. For a subword $u \in \text{Sub}(w)$, a return word for $u$ in $w$ is a subword $r \in \text{Sub}(w)$ having $u$ as a proper prefix, having $u$ as a proper suffix, and having no other occurrences of $u$. The recurrence function $R_w(n)$ is the minimum length such that every factor of $w$ of this length contains every subword in $\text{Sub}_w(n)$. For example, if $w = 001000\diamond010\diamond1\cdots$, then the subword 01010, compatible with the factor 0\diamond010, and the subword 010010 compatible with the factor 010\diamond10, are return words for $u = 010$. Moreover,
\( R_w(2) \geq 12 \) since the prefix 001000\( \cdot \)010\( \cdot \)1 is the shortest prefix of \( w \) having 00, 01, 10, 11 as subwords. If \( R_w(n) \) is finite for all \( n \), we say that \( w \) is uniformly recurrent. A uniformly recurrent partial word is clearly recurrent. It is known that Sturmian full words are uniformly recurrent [5].

If \( w \) is recurrent, the recurrence quotient of \( w \) is defined to be \( \rho(w) = \limsup \frac{R_w(n)}{n} \). In the case of Sturmian full words, computing \( \rho \) can be done using continued fractions [9].

Let \( p \) be a positive integer. We say that an infinite partial word \( w \) is \( p \)-periodic (resp., \( p \)-factor periodic) if \( w_i \uparrow w_j \) (resp., \( w_i = w_j \)) whenever \( i \equiv j \) mod \( p \). If this holds for some \( p \), we say \( w \) is periodic (resp., factor periodic). For example, \( w_1 = (0\cdot001\cdot010)^\omega \) is 3-periodic, but not 3-factor periodic, while \( w_2 = (01\cdot01\cdot01\cdot0)^\omega \) is 3-periodic, but also 3-factor periodic.

We say that \( w \) is ultimately \( p \)-periodic (resp., ultimately \( p \)-factor periodic) if some suffix of \( w \) is \( p \)-periodic (resp., \( p \)-factor periodic), and we also say that \( w \) is ultimately periodic (resp., ultimately factor periodic) if some suffix of \( w \) is periodic (resp., factor periodic). For example, \( w_3 = 0111(0\cdot001\cdot010)^\omega \) is ultimately 3-periodic, but not ultimately 3-factor periodic, while \( w_4 = 0111(01\cdot01\cdot01\cdot0)^\omega \) is ultimately 3-periodic, but also ultimately 3-factor periodic. If \( w \) is an ultimately periodic full word, then \( w = xy^\omega = xyyy\cdots \) for some finite words \( x, y \) with \( y \neq \varepsilon \) called a period of \( w \) (we also call the length \( |y| \) a period). If \( |x| \) and \( |y| \) are as small as possible, then \( y \) is called the minimal period of \( w \). By a well-known theorem of Morse and Hedlund [9], a full word is ultimately periodic if and only if it has bounded subword complexity, that is, the number of length \( n \) subwords is constant, for all sufficiently large \( n \). In [6], this result is explored in the context of partial words.

In addition, we would like to mention that for uniformly recurrent full words, the notions of periodicity and ultimate periodicity are equivalent. This can help the readers understand the relationship between results on full words and partial words.

3 Spectrum of Recurrence Quotients for Partial Words

In this section, we determine the possible values for the recurrence quotients of infinite partial words. In Section 3.1 we study lower bounds on the possible values, while in Section 3.2 we give explicit constructions achieving all possible values. First, examples of infinite periodic partial words with recurrence quotient \( \rho \) in \([2, \infty] \) are provided, and then a method to construct,
from these infinite periodic partial words, nonperiodic infinite partial words achieving these same recurrence quotients. Hence, in the partial word case, the notion of periodicity is irrelevant. (Recall that for full words, periodic and nonperiodic words achieved different recurrence quotients.)

3.1 Lower bounds

Recall that by definition of uniformly recurrent partial word, if a partial word is not uniformly recurrent then its recurrence quotient is infinite.

Also recall that for periodic full words, the recurrence quotient is always 1. In the case of partial words, this is true if we restrict ourselves to ultimately factor periodic words.

Lemma 1. Let \( w \) be a uniformly recurrent infinite partial word that is ultimately factor periodic. Then \( \rho(w) = 1 \).

Proof. Let \( x, y \) be finite partial words with \( w = xy^\omega \). Because \( w \) is recurrent, \( \text{Sub}(w) = \text{Sub}(y^\omega) \).

It is clear that any length \( |y| + n - 1 \) factor of \( y^\omega \) contains every subword in \( \text{Sub}_{y^\omega}(n) = \text{Sub}_w(n) \). But then any length \( |x| + |y| + n - 1 \) factor of \( w \) contains a length \( |y| + n - 1 \) factor of \( y^\omega \), and hence contains every subword in \( \text{Sub}_w(n) \). It follows that \( n \leq R_w(n) \leq |x| + |y| + n - 1 \), so \( \rho(w) = 1 \). \( \square \)

Note that the relation \( R_w(n) \geq p_w(n) + n - 1 \) is no longer valid when considering partial words. This justifies a “long” proof for the following result.

Lemma 2. Let \( w \) be a uniformly recurrent infinite partial word that is not ultimately factor periodic. Then \( \rho(w) \geq 2 \).

Proof. Let \( w \) be a uniformly recurrent partial word that is not ultimately factor periodic. For \( a \in A \), let \( w^a \) be the full word obtained by replacing every \( \diamond \) in \( w \) with \( a \).

We claim that \( w^a \) is not ultimately periodic for some \( a \in A \). Suppose to the contrary; then \( w^0 \) and \( w^1 \) are ultimately periodic, say, with periods \( p \) and \( q \) respectively. Then \( w^0 \) and \( w^1 \) are both ultimately \( pq \)-periodic. Now, for \( b \in A \setminus \{0\} \), \( w^0_i = b \) if and only if \( w_i = b \), so looking at \( w^0 \) we see that every letter \( b \) aside from 0 is such that there exists an integer \( m_b \) such that for \( j > m_b \), \( w_j = b \) if and only if \( w_{j+pq} = b \). But the same argument applies to \( w^1 \), and so there exists an integer \( m_0 \) such that for \( j > m_0 \), \( w_j = 0 \) if and only if \( w_{j+pq} = 0 \). Finally, because \( w \) can be regarded as a full word over
the alphabet $A \cup \{\phi\}$, we see that $w$ is in fact ultimately \(pq\)-factor periodic, a contradiction.

Say without loss of generality that $w^0$ is not ultimately periodic. If $R_w(n) \geq 2n$ for every \(n\), then $\rho(w) \geq 2$ and we are done.

Now suppose $R_w(n_0) < 2n_0$ for some $n_0$. Let $z$ be an element of $\text{Sub}_w(n_0)$ with a maximum number of 0’s. Then every occurrence of $z$ in $w$ must also correspond to an occurrence in $w^0$. It follows from the inequality $R_w(n_0) < 2n_0$ that every return word for $z$ in $w^0$ has length at most $2n_0$. This implies that every subword $u \in \text{Sub}_{w^0}(n)$ of length at least $2n_0$ is composed of (possibly overlapping) occurrences of $z$, with the exception that it may have a prefix that is a suffix of $z$ and a suffix that is a prefix of $z$.

Let $\overline{u}$ be the shortest subword of $w^0$ containing an occurrence of $u$ so that $\overline{u}$ has $z$ as both a prefix and a suffix. Note that $|\overline{u}| \leq n + 2n_0$, and $\overline{u}$ is comprised entirely of (possibly overlapping) occurrences of $z$.

We have $\overline{u} \in \text{Sub}(w^0) \subset \text{Sub}(w)$, so every length $R_w(n + 2n_0)$ factor of $w$ contains $\overline{u}$ as a subword. Because every occurrence of $z$ in $w$ matches an occurrence of $z$ in $w^0$, every occurrence of $\overline{u}$ in $w$ matches an occurrence of $\overline{u}$ in $w^0$. Hence every length $R_w(n + 2n_0)$ factor of $w^0$ contains $\overline{u}$ as a subword. This holds for every $u$ and $\overline{u}$, so $R_w(n) \leq R_w(n + 2n_0)$ for \(n \geq 2n_0\).

It follows that $R_w(n) \geq R_{w^0}(n - 2n_0) \geq 2n - 4n_0$ eventually, and hence $\rho(w) \geq 2$. \(\square\)

Obviously, every partial word is either ultimately factor periodic or not; hence Lemma 1 and Lemma 2 together show that $\mathcal{S}_\phi \subset \{1\} \cup [2, \infty]$.

### 3.2 Constructions

As noted in Lemma 1, every ultimately factor periodic partial word (e.g. $a^\omega$) achieves a recurrence quotient of 1. As proved in Lemma 2, every nonultimately factor periodic partial word achieves a recurrence quotient of 2 or greater. Is every real number greater than or equal to 2 a possible value for being the recurrence quotient of an infinite partial word? In this section, we answer this question affirmatively by constructing infinite partial words with recurrence quotient $\rho$, for each $2 \leq \rho \leq \infty$.

We start with a definition.

**Definition 1.** Let $f(n)$ be a positive integer valued function and let $w$ be an infinite full word. Then we define

$$w^f = \prod_{n=0}^{\infty} w_n^{f(n)-1}.$$ (7)
We call $f$ the chain function for $w^f$.

We first treat the case where $2 < \rho < \infty$.

**Lemma 3.** For any $2 < \beta < \infty$, there exists an infinite partial word with recurrence quotient $\rho = \beta$.

**Proof.** Set $\gamma = \frac{\beta - 1}{\beta - 2} > 1$ and let $w = (0^\omega)^f$, where $f(k) = \lceil \gamma^k \rceil$. For $n \in \mathbb{N}$, let $k_n$ be the largest integer such that $f(k_n) \leq n$. Finally, write $s_k = \sum_{i=0}^{k} f(i) \sim \frac{\gamma^{k+1} - 1}{\gamma - 1}$.

Note that $1^n \in \text{Sub}(w)$ for every $n$ and that it is compatible only with the factor $\diamond^n$. Therefore, a factor of $w$ contains every length $n$ subword of $w$ if and only if it contains $\diamond^n$.

What are the non extendable factors $u$ of $w$ which do not contain $\diamond^n$? We see that either $u$ is a prefix $0_{f(0)}^{-1}0_{f(1)}^{-1} \cdots 0_{f(k_n)}^{-1}0\diamond^{n-1}$ or $u = \diamond^{n-1}0\diamond^{n-1}$. Therefore, $R_w(n) = \max\{s_{k_n} + n + 1, 2n\}$. It follows that

$$\rho(w) = \max \left\{ \limsup \frac{s_{k_n}}{n} + 1 + \frac{1}{n}, 2 \right\} = \max \left\{ \limsup \frac{s_k}{f(k)} + 1, 2 \right\} = \beta,$$

where in the first equality we have used the fact that the lim sup of a maximum is the maximum of the lim supers, and in the second we have used that the lim sup of a sequence is the limit of the sequence of local maximums (provided there are infinitely many).

Slight modifications allow us to achieve $\rho = 2$ and $\rho = \infty$.

**Lemma 4.** There exists an infinite partial word with recurrence quotient $\rho = 2$.

**Proof.** Let $f(k) = 2^k$ and let $w = (0^\omega)^f$. Using the same notation as in the previous proof, we see that $s_k = \sum_{i=0}^{k} 2^{2^i} \sim 2^{2^k} = f(k)$, so

$$\rho(w) = \max \left\{ \lim \frac{s_k}{f(k)} + 1, 2 \right\} = 2.$$

**Lemma 5.** There exists an infinite partial word with recurrence quotient $\rho = \infty$. 

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Proof. Let \( f(0) = 1 \) and \( f(k) = k \) for \( k > 0 \), and let \( w = (0^\omega)^f \). Using the same notation as in the previous proofs, we see that \( s_k = 1 + \sum_{i=1}^{k} i = 1 + \frac{1}{2}(k^2 + k) \gg f(k) \), so

\[
\rho(w) = \max \left\{ \lim \frac{s_k}{f(k)} + 1, 2 \right\} = \infty.
\]

\( \square \)

Lemmas 3, 4, and 5 together with Lemma 1 show that \( \mathcal{S}_c \supset \{1\} \cup [2, \infty] \). With the earlier \( \mathcal{S}_c \subset \{1\} \cup [2, \infty] \) from the end of Section 3.1, this proves the equality \( \mathcal{S}_c = \{1\} \cup [2, \infty] \) in Theorem 2.

Without difficulty, we again modify the constructions to obtain nonultimately periodic words achieving the same recurrence quotients.

**Theorem 3.** If \( w \) is a nonultimately periodic infinite partial word, then \( \rho(w) \in [2, \infty] \).

**Proof.** First, let \( f(k) \) and \( w = (0^\omega)^f \) be defined as in lemma 3 or 4, and let \( g(k) \) be the smallest multiple of \( k \) greater than or equal to \( f(k) \) (take \( g(0) = f(0) \)). Note that \( g(k) \leq f(k) + k \), so \( t_k = \sum_{i=0}^{k} g(i) \leq s_k + O(k^2) \). Let \( v = ((01)^\omega)^g \). Then \( v \) is not ultimately periodic, because for each \( p \geq 1 \) there are infinitely many occurrences of 0 and 1 (in either order) separated by \( jp - 1 \) holes for some \( j \).

Using the same arguments, we have

\[
\rho(v) = \max \left\{ \lim \frac{t_k}{f(k)} + 1, 2 \right\} = \max \left\{ \lim \frac{s_k + O(k^2)}{f(k)} + 1, 2 \right\} = \rho(w).
\]

Of course, the \( \frac{O(k^2)}{f(k)} \) term vanishes in the limit because \( f(k) \) in each case is (at least) exponential.

To modify the construction of Lemma 5, all that is needed is to take the word \( v = ((01)^\omega)^g \) (as \( k \) already divides \( f(k) \)). \( \square \)

## 4 Properties of Recurrent Partial Words

In this section we study properties of recurrent partial words. First we define the hole function \( H \) and the gap function \( h \).

**Definition 2.** Let \( w \) be an infinite partial word. The hole function \( H : \mathbb{N}^+ \to \mathbb{N}^+ \) of \( w \) is such that \( w_i = \diamond \) if and only if \( i = H(m) - 1 \) for some \( m \in \mathbb{N}^+ \). In other words, \( H(m) - 1 \) is the index of the \( m \)th hole in \( w \). The
gap function $h : \mathbb{N}^+ \to \mathbb{N}^+$ of $w$ is the distance between consecutive holes, that is, for each $m \in \mathbb{N}^+$,

$$h(m) = H(m + 1) - H(m).$$

If $w$ has only $M < \infty$ holes, then we take $h(m) = \infty$ for $m \geq M$.

We now examine some relations between recurrence and periodicity.

**Theorem 4.** If $w$ is a recurrent infinite partial word containing at least one hole with gap function $h$ satisfying \( \lim \inf h = \infty \), then $w$ is not ultimately periodic.

**Proof.** Let $w$ be recurrent. Suppose to the contrary that $w$ is ultimately periodic with minimal period $p$. Write $w = x y_1 y_2 \cdots$ with all the $y_i$’s compatible with some full word $\hat{y}$ of length $p$.

First, suppose that $w$ has only one hole. Writing $w = u \diamond v$, recurrence implies that (as we work with an alphabet of at least two letters) any suffix of $v$ has at least one left special factor for each length, thus $v$ cannot be ultimately periodic.

Otherwise, suppose that $w$ has at least two holes. Let $l < l'$ be the positions of any two holes in $w$. Eventually, $h > 3p$ so that we can find two consecutive factors $y_j, y_{j+1}$ beyond $l, l'$ that are full. Write $|x| = q|\hat{y}| + r$ with $0 \leq r < p$ and let $\text{suf}_r(\hat{y})$ be the suffix of $\hat{y}$ of length $r$. Put $\hat{v'} = \text{suf}_r(\hat{y})^{q+r+1}$. Take a completion $\hat{v}$ of $x y_1 \cdots y_j y_{j+1}$ such that $\hat{v}_l \neq \hat{v'}_l$ and $\hat{v}_r \neq \hat{v'}_r$.

Because $\lim \inf h = \infty$, there exists an integer $M$ such that $h(m) > \max(l' - l, 3p)$ for $m > M$. The recurrence of $w$ implies that $\hat{v}$ appears somewhere at some position $i$, that is, $w_i \cdots w_{i+|\hat{y}| - 1} \uparrow \hat{v}$. Moreover, $i$ can be chosen in such a way that the smallest integer $i'$ that satisfies $i \leq H(i') - 1$ also satisfies $i' > M$. We claim that $i \not\equiv 0 \mod p$. For suppose it were. Then $w_i \cdots w_{i+|\hat{y}| - 1} \uparrow \hat{v'}$. Now because $h(m) > l' - l$ for $m > M$, at least one of $w_{i+l}$ and $w_{i+l'}$ is a letter; say without loss of generality, that $w_{i+l}$ is a letter. But then $\hat{v}_l = w_{i+l} = \hat{v}'_l$, a contradiction.

This shows that the $y$-factors of $\hat{v}$ are not aligned with the $y$-factors of $w$. Hence

$$y_j y_{j+1} \uparrow z_1 z_2 z_3 z_4$$

for some nonempty $z_i$’s, with

$$z_2 z_3 \uparrow \hat{y}, \quad z_2 z_1 \uparrow \hat{y}, \quad z_4 z_3 \uparrow \hat{y}, \quad 10$$
and \( z_1 z_2 z_3 z_4 \) having at most one hole (because \( h(m) > 3p \) for \( m > M \)).
That is, \( z_2 z_3 \) is (compatible with) a nontrivial factorization of \( \hat{y} \), \( z_1 \) is the suffix of length \( |z_3| \) of \( \hat{y} \), and \( z_4 \) is the prefix of length \( |z_2| \) of \( \hat{y} \).

Suppose without loss of generality that \( z_1 z_2 \) is full. Then

\[
z_1 z_2 = y_j = \hat{y} = z_2 z_1,
\]

and a well-known result on full words shows that \( \hat{y} \) is a power of a shorter word. This contradicts the minimality of \( p \), so \( w \) cannot be ultimately periodic.

Note that this condition is tight, because the fixed point on \( \diamond \) of the morphism

\[
\varphi(x) = \begin{cases} 
  \diamond ba & \text{if } x = \diamond \\
  ab & \text{if } x \in \{a, b\}
\end{cases}
\]

is recurrent, periodic, has finite \( \liminf h \) and infinite \( \limsup h \). However, this word is not uniformly recurrent, and the following result shows that if we insist on uniform recurrence, then an infinite limit superior is enough to prevent periodicity.

**Theorem 5.** If \( w \) is a uniformly recurrent infinite partial word with gap function \( h \) satisfying \( \limsup h = \infty \), then \( w \) is not ultimately periodic.

**Proof.** The proof is similar to that of Theorem 4, so we only provide a sketch. Suppose to the contrary, that \( w \) is ultimately periodic with minimal period \( p \). Write \( w = xy_1 y_2 \cdots \) with \( |y_i| = p \) and \( y_i \uparrow \hat{y} \) for some full word \( \hat{y} \). Then there exists a prefix \( u \) of \( w \) with at least one hole and that contains a full factor \( y_i y_{i+1} \).

Let \( \hat{u} \) be a completion of \( u \) such that \( u \)'s \( y \)-factors have completions that are incompatible with \( \hat{y} \). Now find a full factor \( v \) of \( w \) of length \( R_w(|u|) \). The subword \( \hat{u} \) must appear somewhere in \( v \), but the \( y \)-factors of \( \hat{u} \) cannot be aligned with those of \( v \). Hence \( \hat{y} \) is a power of some shorter word, which contradicts the minimality of \( p \). \( \Box \)

We next examine a relation between uniform recurrence and subword complexity which states that a uniformly recurrent partial word achieves maximum complexity if and only if its chain function is unbounded.

**Proposition 1.** The subword complexity of a uniformly recurrent infinite partial word is maximal if and only if the number of consecutive holes in this word is unbounded.
Proof. The if direction is obvious.

Let $w$ be a uniformly recurrent infinite partial word. Suppose there exists $C < \infty$ so that beyond some integer $n_0$, the number of consecutive holes is always less than $C$. Suppose to the contrary, that $w$ has maximum subword complexity. Then $a^{R_w(C)}$ appears somewhere beyond $n_0$, compatible with a factor $u$ say. But it is impossible that $b^C$ be a subword of $u$, because $\diamond^C$ does not occur in $u$ and every symbol in $u$ other than $\diamond$ is $a$. But $b^C \in \text{Sub}_w(C)$ and $u$, as a factor of $w$ of length $R_w(C)$, should contain every subword in $\text{Sub}_w(C)$. This is a contradiction; hence $w$ cannot have maximum complexity.

5 Conclusion

As stated earlier, an infinite partial word $w$ is uniformly recurrent if every factor of minimum length $R_w(n)$ contains every subword of length $n$. In this paper we studied the function $R_w(n)$, called the recurrence function of $w$, especially for those partial words $w$ whose recurrence functions do not grow too quickly (e.g. sublinearly). We managed to prove that the recurrence quotient $\rho(w) = \limsup \frac{R_w(n)}{n}$ of any infinite partial word $w$ is 1 or greater than or equal to 2, we determined precisely the spectrum of values in this context, and we gave constructions for each value.

Observe that the proof of Lemma 2 also shows the following result.

Proposition 2. Let $w$ be a uniformly recurrent infinite partial word that is not ultimately factor periodic. Then $R_w(n) \geq 2n - C_w$ ultimately, for some constant $C_w$ that depends on $w$.

For nonperiodic recurrent infinite full words $w$, $R_w(n) \geq 2n$. We conjecture that the same is true for infinite partial words with respect to ultimate factor periodicity.

Conjecture 1. Let $w$ be a uniformly recurrent infinite partial word that is not ultimately factor periodic. Then $R_w(n) \geq 2n$.

A proof of this conjecture would allow us to immediately deduce Lemma 2. However, we can prove the following two weaker results.

Proposition 3. Let $w$ be a uniformly recurrent infinite partial word that is not ultimately factor periodic. Then $R_w(n) \geq 2n$ infinitely often.

Proof. The proof of Lemma 2 shows that $R_w(n) \geq R_w^0(n - 2n_0)$ eventually, assuming, without loss of generality, that $w^0$ is not ultimately periodic and
also assuming that $R_w(n_0) < 2n_0$ for some $n_0$. On the other hand, Theorem 1 states that for the nonperiodic full word $w^0$, $\rho(w^0) \geq 3$. So $R_{w^0}(n) \geq 3n$ infinitely often and thus $R_w(n) \geq R_{w^0}(n - 2n_0) \geq 3(n - 2n_0) \geq 2n$ infinitely often.

**Proposition 4.** Let $w$ be a uniformly recurrent infinite partial word that is not ultimately factor periodic. Then $R_w(n) > \frac{3n}{2}$.

*Proof.* As in the proof of Lemma 2, we can assume that $w^0$ is not ultimately periodic. Suppose there exists $n_0$ such that $R_w(n_0) \leq \frac{3n_0}{2}$. Let $z$ be an element of $\text{Sub}_w(n_0)$ with a maximal number of 0’s. Then every occurrence of $z$ in $w$ must also correspond to an occurrence in $w^0$. It follows from the inequality $R_w(n_0) \leq \frac{3n_0}{2}$ that every return word for $z$ in $w^0$ has length at most $\frac{3n_0}{2}$. So any return word for $z$ is of the form $z_0z$, where $z_0$ is a prefix of $z$, and is also of the form $zz'_0$, where $z'_0$ is a suffix of $z$. Hence we can see that $z$ can be written as $\overline{z}_0^iz_0^i$, where $i$ is a nonnegative integer and $z_0^i$ is a prefix of $z_0$. Since $|z_0| \leq \frac{3n_0}{2}$, $i \geq 2$. Let $p$ be minimal such that $z$ can be written as $\overline{z}_1^iz_10$ for some length $p$ word $z_1$, some prefix $z_10$ of $z_1$, and some integer $j \geq 2$. If the index of the $z$ suffix in a return word is not a multiple of $p$, we have that $z_1 = zz_2z_3 = z_3z_2$ for some full words $z_2$ and $z_3$. A well-known result on full words states that then $z_1$ is a power of a shorter word, a contradiction. Thus $w^0$ is ultimately periodic with period $z_1$, a contradiction. \qed

Finally, let us mention that a World Wide Web server interface at

http://www.uncg.edu/cmp/research/subwordcomplexity5

has been established for automated use of a program that when given as input an infinite partial word $w$ which is a fixed point of a morphism or a periodic word, computes among others the recurrence function $R_w(n)$ and the subword complexity function $p_w(n)$.

**References**


