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- Computational Aspects of Buildings
- June 24–28, 2019
- Accepting graduate student applications soon
On the growth of torsion in the cohomology of arithmetic groups

Dan Yasaki
(joint work with Ash, McConnell, Gunnells)

Department of Mathematics and Statistics
The University of North Carolina at Greensboro

December 8–9, 2018
PANTS XXXI, University of South Carolina
Some explicit $\delta = 1, 2$ computations

Dan Yasaki (joint work with P. Gunnells)

Department of Mathematics and Statistics
The University of North Carolina at Greensboro

September 15, 2012
PANTS XVIII, Wake Forest University
Elstrodt-Grünewald-Mennicke (1982) observed relationships between torsion in abelianization of congruence subgroups of $\text{PSL}_2(\mathbb{Z}[\sqrt{-1}])$ and Galois extensions of $\mathbb{Q}(\sqrt{-1})$. 
Ash (1992) conjectured for \( \Gamma \subseteq \text{SL}_n(\mathbb{Z}) \), every Hecke eigenclass \( \xi \in H^*(\Gamma, \mathbb{F}_p) \) should be attached to a Galois representation \( \rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_n(\mathbb{F}_p) \).
Scholze (2015) proved Ash’s conjecture for a larger class of $\Gamma$. 

Peter Scholze
Nicolas Bergeron  
Akshay Venkatesh  
Haluk Sengün

Bergeron-Venkatesh (2013) and Bergeron-Venkatesh-Sengün (2016) predict surprising torsion growth in the homology groups for certain families of $\Gamma$. 
Scope of investigation and Voronoi homology

Modulo torsion primes,

\[ H^k(\Gamma) \cong H_{\dim(D) - k}(\text{Vor}(\Gamma)). \]

The deficiency \( \delta \) controls the growth,

\[ \delta = \text{rank}_\mathbb{C}(G) - \text{rank}_\mathbb{C}(K). \]

<table>
<thead>
<tr>
<th>( \Gamma )</th>
<th>( \delta )</th>
<th>( \text{dim}_\mathbb{R}(D) )</th>
<th>torsion primes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{GL}_2(\mathcal{O}_L) )</td>
<td>1</td>
<td>3</td>
<td>2, 3</td>
</tr>
<tr>
<td>( \text{GL}_3(\mathbb{Z}) )</td>
<td>1</td>
<td>5</td>
<td>2, 3</td>
</tr>
<tr>
<td>( \text{GL}_2(\mathcal{O}_F) )</td>
<td>1</td>
<td>2 + 3 + 1 = 6</td>
<td>2, 3</td>
</tr>
<tr>
<td>( \text{GL}_4(\mathbb{Z}) )</td>
<td>1</td>
<td>9</td>
<td>2, 3, 5</td>
</tr>
<tr>
<td>( \text{GL}_2(\mathcal{O}_E) )</td>
<td>2</td>
<td>3 + 3 + 1 = 7</td>
<td>2, 3, 5</td>
</tr>
<tr>
<td>( \text{GL}_5(\mathbb{Z}) )</td>
<td>2</td>
<td>14</td>
<td>2, 3, 5</td>
</tr>
<tr>
<td>( \text{GL}_6(\mathbb{Z}) )</td>
<td>2</td>
<td>20</td>
<td>2, 3, 5, 7</td>
</tr>
</tbody>
</table>
Let $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ be a decreasing family of cocompact congruence subgroups with $\cap_k \Gamma_k = \{1\}$. Then

$$\lim_{k \to \infty} \frac{\log |H_i(\Gamma_k; \mathcal{L})_{\text{tors}}|}{[\Gamma : \Gamma_k]}$$

exists for each $i$ and is zero unless $\delta = 1$ and $i = (d - 1)/2$, where $d = \dim D$. In that case, it is strictly positive and equals an explicit constant $c_{G,\mathcal{L}}$ times the volume of $\Gamma \backslash D$.

- $\delta = 0$: little torsion, lots of free
- $\delta = 1$: lots of torsion, little free
- $\delta \geq 2$: relatively little torsion or free
Dan Yasaki (joint work with Ash, McConnell, Gunnells)  On the growth of torsion 12/27

<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>$c_G \ vol(\Gamma \backslash D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$GL_2(\mathcal{O}_L)$</td>
<td>$\frac{</td>
</tr>
<tr>
<td>$GL_2(\mathcal{O}_F)$</td>
<td>$\frac{23^{3/2} \text{reg}_F}{48\pi^5} \zeta_F(2) \approx 0.00234390056921788256219\ldots$</td>
</tr>
<tr>
<td>$GL_3(\mathbb{Z})$</td>
<td>$\frac{\sqrt{3}}{288\pi^2} \zeta(3) \approx 0.00073247603662800481419\ldots$</td>
</tr>
<tr>
<td>$GL_4(\mathbb{Z})$</td>
<td>$\frac{31\sqrt{2}}{259200\pi^2} \zeta(3) \approx 0.00002059998840562887807\ldots$</td>
</tr>
</tbody>
</table>
$L = \mathbb{Q}(\sqrt{-1})$

- $D \simeq \bar{\mathbb{H}}_3$.
- Computations done for $\text{Norm}(\pi) \leq 50000$ (19827 levels).
- Largest torsion at norm 49850, where Voronoi homology is $H_1 = \mathbb{Z}^{18} \times T$,

\[
\# T = 
\begin{align*}
99407444600099014483472905584891296877204680639 \\
86416658793798948901127432947695155728875563424 \\
19476442159847189542963526150932346235466883619 \\
33161406412057509780714570218204049314881664033 \\
9472175527128098186018335659763432414423233944 \\
28888397376030584576028245868131438925540733906 \\
14865670538078059046800867779047996070659056392 \\
69615372231493648172254559736578451714510684160 \\
00000000000.
\end{align*}
\]
Plot with index

Dan Yasaki (joint work with Ash, McConnell, Gunnells)
\[ L = \mathbb{Q}(\sqrt{-3}) \]

- \( D \simeq \mathfrak{A}_3 \).
- Computations done for \( \text{Norm}(\pi) \leq 50000 \) (15294 levels).
- Largest torsion at norm 47604, where Voronoi homology is \( H_1 = \mathbb{Z}^7 \times T \),

\[
\# T = 86458458243032037731955290637022162794763209935 \\
280440087103381631678388632968654056638306185 \\
09382518078533871114953371338042910822277017433 \\
418536350529083146240000000000.
\]
Dan Yasaki (joint work with Ash, McConnell, Gunnells)  
On the growth of torsion
$L = \mathbb{Q}(\sqrt{-15})$

- $D \simeq \mathbb{H}_3$.
- Computations done for $\text{Norm}(\pi) \leq 10103$ (8303 levels).
- Largest torsion at norm 10020, where Voronoi homology is $H_1 = \mathbb{Z}^{142} \times T$.

$$\# T = 41881066680290026290757971072933010839127589372$$
$$20329609346932099823555080316242246455414143824$$
$$81678312213487455195384194363167308898872657519$$
$$11158997541503207392603276379894341069429480519$$
$$65384392910119014805697326867603260287168237074$$
$$47678067481735850787089416159137540458099351433$$

... 12 lines cut ...

$$47612937080789420193496465314969566666312118346$$
$$35590810694991462262604042802662380942618952274$$
$$82502950783747405436363250199487566317958928712$$
$$6121217994412259843396196435033335621142142537$$
$$01636462958029126592626688000000000000000000000000$$
$$00000000000000000000000000000000000000000000000000000000.$$
Dan Yasaki (joint work with Ash, McConnell, Gunnells) On the growth of torsion
$F = \text{cubic field of discriminant } -23$

- $D \simeq \mathbb{H} \times \mathbb{H}_3 \times \mathbb{R}$.
- Computations done for $\text{Norm}(\eta) \leq 5480$ (2011 levels).
Dan Yasaki (joint work with Ash, McConnell, Gunnells)
\( F \) = cubic field of discriminant \(-23\)

- \( D \cong \mathbb{H} \times \mathbb{H}_3 \times \mathbb{R} \).
- Extended computations done for \( H_2 \): \( \text{Norm}(n) \leq 11575 \) (4246 levels).

Largest torsion at norm 10600, where the torsion has size

\[
\# T = 2^3 \cdot 3^{15} \cdot 5^9 \cdot 7^2 \cdot 11 \cdot 103
\]

\[= 12447004217484375000.\]

The largest prime occurs at norm level 11443, where the size of the torsion is exactly the prime contribution 7870506841.

Dan Yasaki (joint work with Ash, McConnell, Gunnells)
Dan Yasaki  (joint work with Ash, McConnell, Gunnells)  On the growth of torsion  22 / 27
\( D \) is 5-dimensional symmetric space.

Computations done for \( \text{Norm}(n) \leq 641 \) (641 levels).

Largest torsion at norm 570, where the size of torsion is

\[
2^{154} \cdot 3^{27} \cdot 5^2 \cdot 7^6 \cdot 11^6 \cdot 17^{10} \cdot 37^6 \cdot 47^2 \cdot 131^2 \cdot 619^6 \cdot 3137^6 \cdot 6113^6 \cdot 2723737^2 \cdot 242222857291^2 \cdot 278917146364629278585122304155523929101710815974757^2.
\]

The largest prime occurs at level 638. The torsion size is

\[
2^{106} \cdot 3^{22} \cdot 5^8 \cdot 7^2 \cdot 11^2 \cdot 31^6 \cdot 8969^6 \cdot 1537835313687313016296678426257182245274808716382 \cdot 19475779265644361^2.
\]
Plot with index

Dan Yasaki (joint work with Ash, McConnell, Gunnells)

On the growth of torsion
$D$ is 9-dimensional symmetric space
Computations done for $\text{Norm}(n) \leq \sim 60 - 120$.
We computed $H_3$ for levels less than or equal to 119. At level 114, we have the largest torsion group in this range,

$$2^{12} \cdot 3^7 \cdot 11^4.$$ 

The largest prime occurs at level 119,

$$2^4 \cdot 3^3 \cdot 31^4.$$
Thank you.