Read all of the following information before starting the exam:

- It is to your advantage to answer ALL of the 11 questions.
- It is your responsibility to make sure that you have all of the problems.
- There is no need to complete the test in order. The problems are independent.
- Correct numerical answers with insufficient justification may receive little or no credit.
- Clearly distinguish your final answer from your scratch work with a box or circle.
- *Budget your time!*
- If you have read all of these instructions, draw a happy face here.
1. (10 points) Use the Intermediate Value Theorem to show that \( f(x) = x^3 - x^2 - 1 \) has a root in the interval \([1, 2]\).

**Solution:** First note that \( f \) is continuous on \([1, 2]\). (It is in fact continuous everywhere.) We have that \( f(1) = -1 \) and \( f(2) = 8 - 4 - 1 = 3 \). Since 0 is between -1 and 3, the Intermediate Value Theorem guarantees the existence of \( c \) between 1 and 2 such that \( f(c) = 0 \).

2. (5 points) (Precise definition of limit) Let \( f(x) \) be defined on an open interval containing \( x_0 \), except possibly at \( x_0 \) itself. We say that the limit of \( f(x) \) as \( x \) approaches \( x_0 \) is \( L \), denoted \( \lim_{x \to x_0} f(x) = L \), if

**Solution:** for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( x \neq x_0 \),

\[
0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.
\]

3. (5 points) Find a number \( \delta > 0 \) such that every number \( x \) in the interval \(|x - 1| < \delta\) also satisfies \(|(8x - 2) - 6| < \frac{1}{100}\).

**Solution:** We have that

\[
|(8x - 2) - 6| = |8x - 8| = 8|x - 1| < \frac{1}{100}
\]

as long as \(|x - 1| < \frac{1}{800}\). Thus we can choose \( \delta = \frac{1}{800} \).
4. Let \( f(x) = \frac{x^2}{2x - 10}. \)
   
   (a) (5 points) Evaluate \( \lim_{x \to 5^-} f(x) \) and \( \lim_{x \to 5^+} f(x). \)

   **Solution:** Note that if we tried to plug in \( x = 5 \), we would get \( \frac{25}{0} \). That means the limit is \( +\infty, -\infty \), or does not exist. For values \( x \) near 5 but less than 5, \( x^2 \) is positive and \( 2x - 10 \) is negative. Thus \( \frac{x^2}{2x - 10} \) is negative for such values. It follows that
   \[
   \lim_{x \to 5^-} \frac{x^2}{2x - 10} = -\infty.
   \]

   For values \( x \) near 5 but greater than 5, \( x^2 \) is positive and \( 2x - 10 \) is positive. Thus \( \frac{x^2}{2x - 10} \) is positive for such values. It follows that
   \[
   \lim_{x \to 5^+} \frac{x^2}{2x - 10} = \infty.
   \]

   (b) (5 points) Does the graph \( y = f(x) \) have a vertical asymptote? If it does, give the formula for the vertical asymptote. If not, explain why not.

   **Solution:** The only possible place this graph can have a vertical asymptote is where \( 2x - 10 = 0 \). That means \( x = 5 \) is the only possibility. Since the numerator is not zero when \( x = 5 \), this shows that \( x = 5 \) is indeed a vertical asymptote.

5. Let \( f(x) = \frac{3x^3 + 2x - 13}{7x^3 + 23x^2 + x - 1}. \)
   
   (a) (5 points) Evaluate \( \lim_{x \to \infty} f(x). \)

   **Solution:** Polynomials are dominated by their leading term. Hence
   \[
   \lim_{x \to \infty} \frac{3x^3 + 2x - 13}{7x^3 + 23x^2 + x - 1} = \lim_{x \to \infty} \frac{3x^3}{7x^3} = \lim_{x \to \infty} \frac{3}{7} = \frac{3}{7}.
   \]

   (b) (5 points) Does the graph \( y = f(x) \) have a horizontal asymptote? If it does, give the formula for the horizontal asymptote.

   **Solution:** Yes. The horizontal asymptote (from the computation above) is \( y = \frac{3}{7} \).
6. Evaluate the following limits

(a) (5 points) \( \lim_{t \to -2} \frac{t + 2}{t^2 + 3t + 2} \)

**Solution:** Notice that if we plug in \( t = -2 \), we would get \( \frac{0}{0} \). That means we need to *do more work*. We compute

\[ \lim_{t \to -2} \frac{t + 2}{t^2 + 3t + 2} = \lim_{t \to -2} \frac{(t + 2)}{(t + 2)(t + 1)} = \lim_{t \to -2} \frac{1}{t + 1} = -1. \]

(b) (5 points) \( \lim_{x \to 0} \frac{\sin(5x)}{3x} \)

**Solution:**

\[ \lim_{x \to 0} \frac{\sin(5x)}{3x} = \frac{1}{3} \lim_{x \to 0} \frac{\sin(5x)}{x} = \frac{1}{3} \lim_{x \to 0} \frac{5\sin(5x)}{5x} = \frac{5}{3} \lim_{x \to 0} \frac{\sin(5x)}{5x}. \]

Let \( \theta = 5x \). As \( x \to 0 \), we have \( \theta \to 0 \). Then

\[ \lim_{x \to 0} \frac{\sin(5x)}{5x} = \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1, \]

so \( \lim_{x \to 0} \frac{\sin(5x)}{3x} = \frac{5}{3} \).

7. (10 points) Suppose \( f \) is a function such that \( \lim_{x \to 1} f(x) = 2 \). Suppose \( g \) is a function such that \( \lim_{x \to 1} g(x) = 4 \). Use Limit Laws to compute \( \lim_{x \to 1} (4f(x) - \sqrt{g(x)}) \).

**Solution:**

\[ \lim_{x \to 1} (4f(x) - \sqrt{g(x)}) = \lim_{x \to 1} 4f(x) - \lim_{x \to 1} \sqrt{g(x)} \]

\[ = 4 \lim_{x \to 1} f(x) - \lim_{x \to 1} \sqrt{g(x)} \]

\[ = 4 \lim_{x \to 1} f(x) - \sqrt{\lim_{x \to 1} g(x)} \]

\[ = 4(2) - \sqrt{4} \]

\[ = 6. \]
8. (10 points) At what points is the function $f(x) = \frac{x + 3}{x^2 - 3x - 10}$ continuous?

**Solution:** Note that $f$ is a rational function, and so it is continuous on its domain. The domain of a rational function is all real numbers, except the roots of the denominator. Since $x^2 - 3x - 10 = (x - 5)(x + 2)$, we have that $f$ is continuous on $(-\infty, -2) \cup (-2, 5) \cup (5, \infty)$.

9. (10 points) For what value of $a$ is

$$f(x) = \begin{cases} 
  x^2 + 1 & \text{if } x < -2, \\
  5ax & \text{if } x \geq -2
\end{cases}$$

continuous at $x = -2$?

**Solution:** We have that $f$ is continuous at $x = 3$ if $\lim_{x \to 3} f(x) = f(3)$. We compute $\lim_{x \to -2^-} f(x) = (-2)^2 + 1 = 5$ and $\lim_{x \to -2^+} f(x) = 5 \cdot a \cdot (-2) = -10a$. It follows that to arrange that $f$ is continuous at $x = 3$, we must have $5 = -10a$. Thus choosing $a = -\frac{1}{2}$ will make $f$ continuous at $x = 2$.
10. (10 points) Suppose \( f \) and \( g \) are continuous functions such that
\[
\lim_{x \to 0} f(x) = 2, \quad f(7) = -1, \quad \lim_{x \to 0} g(x) = 7, \quad \text{and} \quad g(2) = 3.
\]
Compute \( \lim_{x \to 0} g(f(x)) \) or explain what additional information is needed to compute the limit.

**Solution:** Recall that the composition of continuous functions is continuous so that
\[
\lim_{x \to 0} g(f(x)) = g(\lim_{x \to 0} f(x)) = g(2) = 3.
\]

11. The graph of \( y = f(x) \) is shown below. Compute the following or explain why it does not exist.

(a) (2 points) \( \lim_{x \to -2^+} f(x) \)

**Solution:** As we approach \(-2\) from the right, the height of the graph approaches \(-3\), so \( \lim_{x \to -2^+} f(x) = -3 \).

(b) (2 points) \( \lim_{x \to -2^-} f(x) \)

**Solution:** As we approach \(-2\) from the left, the height of the graph approaches \(4\), so \( \lim_{x \to -2^-} f(x) = 4 \).

(c) (2 points) \( f(-2) \)
Solution: The height at $-2$ is $-3$, so $f(-2) = -3$.

(d) (2 points) $\lim_{x \to 4} f(x)$

Solution: As we approach 4 from the left and from the right, the height of the graph approaches 3, so $\lim_{x \to 4} f(x) = 3$.

(e) (2 points) $\lim_{x \to 2} f(x)$

Solution: As we approach 2 from the left and from the right, the height of the graph approaches 1, so $\lim_{x \to 2} f(x) = 1$.