HOMEWORK 1

DAN YASAKI

1. Read Gunnells Modular forms TWIGS. 
   http://www.math.umass.edu/~gunnells/talks/modforms.pdf
2. Read Chapter 1 of textbook.
3. §1.6 (1.1) Note that this shows the action of $GL_2(\mathbb{R})$ preserves the complex upper halfplane.
   **Solution:** Let $z = x + iy \in \mathbb{C}$ with $y > 0$, and let $a, b, c, d \in \mathbb{R}$ with $ad - bc > 0$. We want to show that
   $$\text{Im} \left( \frac{az + b}{cz + d} \right) > 0.$$ 
   Multiply the numerator and denominator by $cz + d = c\bar{z} + d$ to get
   $$\left( \frac{az + b}{cz + d} \right) = \left( \frac{az + b}{cz + d} \right) \left( \frac{c\bar{z} + d}{c\bar{z} + d} \right) = \frac{ac|z|^2 + bcz + adz + bd}{|cz + d|^2} = \frac{ac|z|^2 + bcx - bciy + adx + adiy + bd}{|cz + d|^2}.$$ 
   The imaginary part is $\frac{(ad - bc)y}{|cz + d|^2}$, which is greater than 0 as desired.
4. §1.6 (1.3) 
   **Solution:** Recall a weakly modular function is a meromorphic function such that for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma,$
   $$f(\gamma \cdot z) = (cz + d)^k f(z).$$ 
   (a) Suppose $f$ and $g$ are weakly modular functions of weight $k_1$ and $k_2$, respectively. We want to show the product $h = fg$ is a weakly modular function. The product of meromorphic functions is meromorphic, so it suffices to show that $h$ satisfies the correct equivariance properties. We compute
   $$h(\gamma \cdot z) = f(\gamma \cdot z)g(\gamma \cdot z) = (cz + d)^{k_1} f(z)(cz + d)^{k_2} g(z) = (cz + d)^{k_1 + k_2} f(z)g(z) = (cz + d)^{k_1 + k_2} h(z).$$ 
   (b) Suppose $f$ is a weakly modular function of weight $k$. We want to show that $1/f$ is a weakly modular function. The reciprocal of a meromorphic function is meromorphic, so it suffices to show that $h = 1/f$ satisfies the correct equivariance
properties. We compute
\[
h(\gamma \cdot z) = \frac{1}{f(\gamma \cdot z)}
= \frac{1}{(cz + d)kf(z)}
= (cz + d)^{-k} \frac{1}{f(z)}
= (cz + d)^{-k} h(z).
\]

(c) Suppose \( f \) and \( g \) are modular functions. We want to show that \( fg \) is a modular function. Recall that a modular function is a weakly modular function that is meromorphic at infinity. Above we show that the product of weakly modular functions is weakly modular, so it suffices to show that \( h = fg \) is meromorphic at infinity, assuming \( f \) and \( g \) are meromorphic at infinity. This can be shown by multiplying the respective \( q \) expansions. Specifically, let
\[
f(z) = \sum_{n \geq m_1} a_n q^n \quad \text{and} \quad g(z) = \sum_{n \geq m_2} b_n q^n.
\]
Then the \( q \)-expansion of \( h \) is
\[
h(z) = \left( \sum_{n \geq m_1} a_n q^n \right) \left( \sum_{n \geq m_2} b_n q^n \right)
= a_{m_1} b_{m_2} q^{m_1 + m_2} + \cdots
\]
Since \( m_1 + m_2 \in \mathbb{Z} \), it follows that \( h \) is meromorphic at infinity.

(d) Suppose \( f \) and \( g \) are modular forms. We want to show that \( h = fg \) is a modular form. Recall that a modular form is a modular function that is holomorphic on \( \mathbb{H} \) and holomorphic at infinity. We show above that the product of modular functions is a modular function. The product of holomorphic functions is holomorphic. Thus it suffices to show that \( h \) is holomorphic at infinity assuming \( f \) and \( g \) are holomorphic at infinity. As above, we just look at the \( q \)-expansions. Specifically, let
\[
f(z) = \sum_{n \geq 0} a_n q^n \quad \text{and} \quad g(z) = \sum_{n \geq 0} b_n q^n.
\]
Then the \( q \)-expansion of \( h \) is
\[
h(z) = \left( \sum_{n \geq 0} a_n q^n \right) \left( \sum_{n \geq 0} b_n q^n \right)
= a_0 b_0 + (a_1 b_0 + a_0 b_1) q + \cdots
\]
Since \( m_1 + m_2 \in \mathbb{Z} \), it follows that \( h \) is meromorphic at infinity.

5. §1.6 (1.4)

Solution: Recall
\[
\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}.
\]
(a) Let \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( h = \begin{bmatrix} q & r \\ s & t \end{bmatrix} \) be elements of \( \Gamma_1(N) \). Then
\[
a \equiv d \equiv q \equiv t \equiv 1 \pmod{N}
\]
and
\[
c \equiv s \equiv 0 \pmod{N}.
\]
It follows that
\[
g^{-1} = \begin{bmatrix} d & -b \\ -ca & -b \end{bmatrix} \in \Gamma_1(N).
\]
We compute
\[
gh = \begin{bmatrix} aq + bs & ar + bt \\ qc + ds & cr + dt \end{bmatrix}.
\]
Since \( c \equiv s \equiv 0 \pmod{N} \), we have \( qc + ds \equiv 0 \pmod{N} \). Since \( a \equiv q \equiv 1 \pmod{N} \) and \( s \equiv 0 \pmod{N} \), we have \( aq + bs \equiv 1 \pmod{N} \). Similarly, we have \( cr + dt \equiv 1 \pmod{N} \). Thus \( gh \in \Gamma_1(N) \), and \( \Gamma_1(N) \) is a subgroup of \( \text{SL}_2(\mathbb{Z}) \).

(b) We want to prove that \( \Gamma_1(N) \) has finite index in \( \text{SL}_2(\mathbb{Z}) \), where
\[
\Gamma(N) = \ker(\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/NZ)).
\]
First note that \( \Gamma(N) \subset \Gamma_1(N) \). It follows that
\[
[\text{SL}_2(\mathbb{Z}) : \Gamma_1(N)] \leq [\text{SL}_2(\mathbb{Z}) : \Gamma(N)] \leq \# \text{SL}_2(\mathbb{Z}/NZ) < \infty.
\]
(c) We want to prove that \( \Gamma_0(N) \) has finite index in \( \text{SL}_2(\mathbb{Z}) \). This follows because \( \Gamma_1(N) \subset \Gamma_0(N) \), and we show above that \( \Gamma_1(N) \) has finite index in \( \text{SL}_2(\mathbb{Z}) \).
(d) We want to prove that \( \Gamma_0(N) \) and \( \Gamma_1(N) \) have level \( N \). Recall that the level of a congruence subgroup is the smallest positive integer \( n \) such that the congruence subgroup contains \( \Gamma(n) \). Let \( t < N \). Then \( g = \begin{bmatrix} 1 & 0 \\ t & 0 \end{bmatrix} \in \Gamma(t) \), and \( g \notin \Gamma_1(N) \) and \( g \notin \Gamma_0(N) \). It follows that the level of \( \Gamma_1(N) \) and the level of \( \Gamma_0(N) \) is greater than or equal to \( N \). It is clear that \( \Gamma(N) \subset \Gamma_0(N) \) and \( \Gamma(N) \subset \Gamma_0(N) \), and so the level is less than or equal to \( N \). It follows that the level is exactly \( N \).

6. §1.6 (1.7) Note that this shows that
\[
(f^{[\gamma]k})(z) = \det(\gamma)^{k-1}(cz + d)^{-k}f(\gamma(z))
\]
defines a right action of \( \text{GL}_2(\mathbb{R}) \) on the set of functions \( f : \mathbb{H}^* \to \mathbb{C} \).

**Solution:** For \( \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), let \( j \) be the automorphy factor \( j(\gamma, z) = (cz + d) \). Note that
\[
f^{[\gamma]k}(z) = \det(\gamma)^{k-1}j(\gamma, z)^{-k}f(\gamma \cdot z).
\]
We want to show that
\[
f^{[\gamma_1\gamma_2]k}(z) = (f^{[\gamma_1]k}j^{[\gamma_2]k})(z).
\]
The left side is
\[
\det(\gamma_1\gamma_2)^{k-1}j(\gamma_1\gamma_2, z)^{-k}f((\gamma_1\gamma_2) \cdot z)
\]
and the right side is
\[
\det(\gamma_2)^{k-1}\det(\gamma_1)^{k-1}j(\gamma_1, \gamma_2, z)^{-k}j(\gamma_2, z)^{-k}f((\gamma_1 \cdot (\gamma_2 \cdot z))).
\]
Thus it suffices to show that
(a) $(\gamma_1 \gamma_2) \cdot z = \gamma_1 \cdot (\gamma_2 \cdot z)$ and
(b) $j(\gamma_1 \gamma_2, z) = j(\gamma_1, \gamma_2 \cdot z) j(\gamma_2, z)$.

Consider the vector $\begin{bmatrix} z \\ 1 \end{bmatrix}$. Then one can relate the action of matrices on the upper half plane with the regular matrix multiplication on vectors by

$$\gamma \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma \cdot z \\ 1 \end{bmatrix} j(\gamma, z).$$

It follows that

(1) $$\begin{bmatrix} \gamma_1 \gamma_2 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \gamma_1 \begin{bmatrix} \gamma_2 \cdot z \\ 1 \end{bmatrix} j(\gamma_2, z)$$

(2) $$= \begin{bmatrix} \gamma_1 \cdot (\gamma_2 \cdot z) \\ 1 \end{bmatrix} j(\gamma_1, \gamma_2 \cdot z) j(\gamma_2, z).$$

On the other hand,

(3) $$\begin{bmatrix} \gamma_1 \gamma_2 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} (\gamma_1 \gamma_2) \cdot z \\ 1 \end{bmatrix} j(\gamma_1 \gamma_2, z).$$

Setting (2) equal to (3) gives the desired result.

Dan Yasaki, Department of Mathematics and Statistics, University of North Carolina at Greensboro, Greensboro, NC 27402-6170, USA

E-mail address: d_yasaki@uncg.edu