Abstract. Notes and questions about perfect pairings. This arose in the context of a summer reading course from Stein’s [1].

Let \( R \) be field, and let \( M, N, \) and \( L \) be vector spaces over \( R \). (We will usually consider \( R = \mathbb{Q}, \mathbb{R}, \mathbb{C}, \) or \( \mathbb{F}_p \). Our vector spaces will usually be finite-dimensional.) Many of the things below are true even when \( R \) is a ring in the context of \( R \)-modules.

Exercise 1. \( \text{Hom}_R(M, R) \) is the space of linear functionals on \( M \). It is often denoted \( M^* \), and called the dual space of \( M \). More generally, let \( \text{Hom}_R(M, N) \) denote the set of \( R \)-linear maps from \( M \) to \( N \). Prove \( \text{Hom}_R(M, N) \) is a vector space. Assume that \( M \) and \( N \) are finite dimensional. Compute the dimension of \( \text{Hom}_R(M, N) \).

Definition 1. A \( R \)-bilinear map \( \langle \cdot, \cdot \rangle : M \times N \to L \) is called a pairing.

Exercise 2. A good example to keep in mind is the pairing between \( M^* \) and \( M \). Specifically, define \( \langle \cdot, \cdot \rangle : M^* \times M \to R \) by \( \langle f, m \rangle = f(m) \). Prove that this is in fact a pairing.

Exercise 3. Suppose \( \langle \cdot, \cdot \rangle : M \times N \to L \) is a pairing. We can view \( \langle \cdot, \cdot \rangle \) as a \( R \)-linear map \( \Phi_1 : M \to \text{Hom}_R(N, L) \). We can also view \( \langle \cdot, \cdot \rangle \) as a \( R \)-linear map \( \Phi_2 : N \to \text{Hom}_R(M, L) \). Explain. (Hint for \( \Phi_1 \): Given \( m \in M \), what is the most natural way to get a map from \( N \) to \( L \) using what is given?)

Definition 2. A pairing is non-degenerate if whenever \( \langle m, n \rangle = 0 \) for all \( n \in N \), then \( m = 0 \).

Exercise 4. Explain non-degeneracy in terms of \( \Phi_1 \) or \( \Phi_2 \).

Definition 3. A pairing is perfect if \( \Phi_1 \) is an isomorphism.

Exercise 5. If \( \langle \cdot, \cdot \rangle \) is a perfect pairing of finite-dimensional vectors spaces, is \( \Phi_2 \) is an isomorphism?


Exercise 7. Let \( \langle \cdot, \cdot \rangle \) be the usual inner product on \( \mathbb{R}^n \). Prove that \( \langle \cdot, \cdot \rangle \) is a non-degenerate, perfect pairing.

Exercise 8. For each pair of vectors \( u \) and \( v \) in \( \mathbb{R}^2 \), define \( \langle u, v \rangle \) to be the determinant of the matrix with columns \( u \) and \( v \). Prove \( \langle \cdot, \cdot \rangle \) is a pairing. Is it nondegenerate? Is it perfect?

Exercise 9. For each \( A \in \text{Mat}_n(R) \) and each \( v \in R^n \), define \( \langle A, v \rangle = Av \). Is this a pairing? Is it nondegenerate? Is it perfect?

Exercise 10. For each \( f, g \in C^\infty(\mathbb{R}) \), define

\[
\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx.
\]

Is this a pairing?
Exercise 11. For each $f \in C^\infty(\mathbb{R})$ and each closed interval $[a, b] \subset \mathbb{R}$, define

$$\langle f, [a, b] \rangle = \int_a^b f(x) \, dx.$$ 

Is this a pairing? Before answering that, think carefully about what you would need to show. What is $M$, $N$, and $L$ in this case?

References