1. Introduction

The purpose of this article is to provide a short, self-contained exposition on the natural exponential function e^x starting from an accessible definition to the derivation of all of its properties. The natural exponential function is perhaps the most important function in mathematics. Its applications range from mathematics, statistics, and economics to physics and other natural sciences. Possibly because of its wide use and importance, there is a variety of available definitions and approaches to the function. We will discuss the most common definitions, show their strengths and weaknesses and at the end we will select one unified approach that we think is the most appropriate for mathematicians as well as those who only need to learn the basics about e^x.

Wikipedia, see reference 9, lists the following five definitions at the time of writing:

(D1) \( e^x := \lim_{n \to \infty} (1 + x/n)^n \).

(D2) \( e^x \) is the inverse of \( \ln x := \int_1^x (1/s) \, ds \).

(D3) \( e^x := \sum_{n=0}^{\infty} x^n / n! \).

(D4) \( e^x \) is the only continuous function \( f(x) \) satisfying \( f(a+b) = f(a) f(b) \) for all \( a, b \in \mathbb{R} \) and \( f(1) = \sum_{n=0}^{\infty} 1/n! \).

(D5) \( e^x \) is the solution of the differential equation \( y' = y \) satisfying \( y(0) = 1 \).

Every single one of the above definitions constitutes one possible approach to the function. From a mathematical point of view, all definitions define the same function and, hence, it does not really matter which one we choose. However, choosing the 'right' approach can significantly affect and potentially improve our understanding of the function and mathematics in general.

We have to say right away that there is, very likely, no best approach. As we will discuss in the next section, each definition has its strengths and weaknesses. What seems to be right for one person may sound completely outrageous for another. Ultimately, the best approach will vary based on the circumstances. The current article argues that if somebody does not know much mathematics, yet for various reasons needs to learn and understand a great deal about e^x because of its applications in economics and the natural sciences, then definition (D5) together with the mathematical methods implicitly contained in this definition are the best way to start.

2. Strengths and weaknesses of different definitions

Definition (D1) contains the most primitive terms. Theoretically, we can understand this definition with minimal mathematical knowledge and background. Another advantage is that
this definition of \( e^x \) can be naturally motivated by the example of compounding interests. The drawback of this definition is that the derivation of other properties from it is technical; see reference 4, pp. 51 and 133.

In calculus classes, the exponential function is usually defined by (D2); see, e.g. references 1 (p. 428), 7 (p. 425), and 8 (p. 331). Compared to the definition (D1), this approach requires nontrivial knowledge and understanding of definite integrals. Most importantly, there does not seem to be a natural motivation for this definition. Probably the only reason why it is so widely spread is that it fits the current curriculum with the least resistance.

Definition (D3) and the approach to functions through series is universal. One can define other functions in the same way (for example, \( \sin x = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)! \), or \( \cos x = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)! \)). Moreover, the formula can also be used to define exponentials of matrices (and linear operators in general); see reference 3. These are the reasons why this definition is often used in real and functional analysis; see reference 6, p. 178. However, the formula requires a good knowledge of series and, thus, this definition is not suitable for relative beginners in mathematics. Also, despite its universality, the formula by itself does not provide sufficient motivation.

The property \( e^{a+b} = e^a e^b \), which is the core of definition (D4), is the reason why \( e^x \) is an important function. Definition (D4) is the most abstract yet the most beautiful definition of \( e^x \). We will see below how this property demonstrates itself by the lack of memory in many natural processes. More importantly, this definition is elegant and its beauty (the mere fact that it is equivalent to all other definitions is beautiful) constitutes the essence of mathematics.

If the authors were not primarily interested in a simple use of \( e^x \) in biology and other natural sciences through calculus, but instead aimed for deeper applications in probability and mathematical modelling, they would argue that (D4) is the best way to approach \( e^x \). However, for the rest of the article we will advocate why one should choose (D5) for the definition of the exponential function.

### 3. \( e^x \) as a unique solution of \( y' = y \), \( y(0) = 1 \)

Let us start from the beginning.

**Definition 1** The natural exponential function \( e^x \) is defined to be the only function \( y = y(x) \) that satisfies the following two conditions:

- (E1) \( y'(x) = y(x) \) for all \( x \in (-\infty, \infty) \), and
- (E2) \( y(0) = 1 \).

The definition does not really say that such a function exists and is unique. Similar problems arise with other definitions of \( e^x \), yet they can be fixed in a relatively elementary way. Here one needs an advanced tool, namely Picard's theorem (see reference 2, p. 110), to guarantee the existence and uniqueness. A much weaker version of the theorem which is enough for our purposes is stated below.

**Theorem 1** *(Picard.)* For any numbers \( k \) and \( c_0 \) there exists a unique function \( y(x) \) satisfying

- \( y'(x) = ky(x) \) for all \( x \in (-\infty, \infty) \), and
- \( y(0) = c_0 \).
We can also argue that, on top of Picard's theorem, we also use differential equations implicitly contained in the definition. Still, this definition is understandable to anybody with the knowledge of a derivative; and there are significant advantages of this definition: (1) a natural motivation and (2) an easy way to derive other properties of \( e^x \).

4. Motivational examples

Let us consider a savings account such that the interest is added to the principal at every moment, and from the moment the interest is added, the account also accrues interest on that interest. Thus, if the interest is 100% per a certain unit of time and \( dx \) denotes a very small portion of time, we have

\[
y(x + dx) = y(x) + y(x) \, dx
\]

for all \( x \), where \( y(x) \) denotes the account balance at time \( x \). This means that

\[
\frac{y(x + dx) - y(x)}{dx} \approx y(x)
\]

and, consequently, it yields our equation \( y'(x) = y(x) \). We note that the above derivation is not rigorous but motivational only.

In general, if \( y(x) \) represents the amount of a certain quantity at time \( x \), the property \( y' = y \) means that the rate of growth is proportional to its size. The savings account example illustrates the common knowledge that money (either in the form of savings or debt) behaves this way. Someone with a background in the natural sciences also thinks of uninhibited growth and/or radioactive decay.

One important feature of a savings account is the lack of the memory within the system. A dollar does not remember whether it was added as a principal or interest and when it happened; it still yields the same amount of interest as it would have if it had been in the principal from the beginning. Also, the future balance of the account depends only on the current balance.

The lack of memory is responsible for the property

\[
e^{a+b} = e^a e^b.
\]

Indeed, for simplicity assume that \( e^a \) is an integer. If we put $1 into a savings account, in time \( x = a \) the balance will be \( e^a \). If we let the account grow for some additional time \( b \) and mentally track every single one of the \( e^a \) dollars, the new balance will be \( e^a e^b \). However, we just had $1 in that account for the total time \( x = a + b \). Thus, the balance is \( e^{a+b} \) and, consequently, \( e^{a+b} = e^a e^b \). The property of being memory-free is illustrated by the fact that one can 'forget' the past by 'restarting' the clock at time \( x = a \).

The example of uninhibited growth of bacteria (see, e.g. reference 1, p. 605) is possibly the best motivation for a person with knowledge of biology. We just have to be careful, because, as always with mathematics, once it starts to touch real life, many of the idealistic assumptions of the mathematical model can easily be violated. To illustrate this point, we need to realize that considering only one bacterium at the beginning is not appropriate, since the system would behave in a discrete way. Starting with \( N \) bacteria, for very large \( N \), and considering any collection of \( N \) bacteria to be 1 (colony) helps to take care of the discreteness (at least for the initial period of time), especially if we assume that the bacteria do not all split at the same time, but rather their splitting time is uniformly distributed. On the other hand, this idealization is not completely without memory, since once a bacterium splits, the two new bacteria are like
identical twins with the same inner clocks; they split together at the same time (after which there would be four identical bacteria, etc.), i.e. the bacteria sort of remember the common ancestors.

5. Properties of $e^x$

**Fact 1** $e^x = \lim_{n \to \infty} (1 + x/n)^n$.

*Proof* Denote $y(x) = \lim_{n \to \infty} (1 + x/n)^n$. Then we have

$$y'(x) = \frac{d}{dx} \left( \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n \right)$$

$$= \lim_{n \to \infty} \frac{d}{dx} \left( 1 + \frac{x}{n} \right)^n$$

$$= \lim_{n \to \infty} n \left( 1 + \frac{x}{n} \right)^{n-1}$$

$$= \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^{n-1}$$

$$= \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^{-1}$$

$$= y(x).$$

The second equality (where we interchanged the limit and differentiation) follows by reference 6, Theorem 7.17. Thus, we have $y'(x) = y(x)$. Moreover,

$$y'(0) = \lim_{n \to \infty} \left( 1 + \frac{0}{n} \right)^n = 1.$$ 

Hence, by the uniqueness of the exponential function, $y(x) = e^x$ which is exactly what we wanted to prove.

**Fact 2** $e^x$ is the inverse of $\ln x := \int_1^x (1/s) \, ds$.

*Proof* Let $y(x)$ denote the inverse of $\ln x$. By the fundamental theorem of calculus (see reference 1, p. 403),

$$\frac{d}{dx} \ln(x) = \frac{1}{x}.$$

Thus, by the theorem on differentiation of inverse functions (see reference 1, p. 249),

$$y'(x) = \frac{1}{1/y(x)} = y(x).$$

Because $\ln 1 = 0$, we get $y(0) = 1$ and, thus, by the uniqueness of the exponential function, $y(x) = e^x$ which we wanted to prove.
**Fact 3** \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \).

**Proof** Denote \( y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \). Then we have

\[
y'(x) = \frac{d}{dx} \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{d}{dx} \left( \frac{x^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = y(x),
\]

where the second equality holds because of the theorem on differentiation of power series; see reference 1, p. 704. Thus, \( y'(x) = y(x) \); and since \( y(0) = \sum_{n=0}^{\infty} 0^n/n! = 1 \), we must have, by the uniqueness of the exponential function, \( y(x) = e^x \).

**Fact 4** \( e^{a+b} = e^a e^b \) for all \( a, b \in (-\infty, \infty) \).

**Proof** Fix \( a \in (-\infty, \infty) \) and consider the function

\[ y(x) = e^{a+x} . \]

Since

\[ y'(x) = e^{a+x} = y(x), \quad y(0) = e^a , \]

the uniqueness property provided by Picard’s theorem yields

\[ y(x) = e^a e^x , \]

since \( e^a e^x \) is another function that shares the above properties. Putting \( x = b \) gives the result.

**Fact 5** The natural exponential function \( e^x \) has the following properties.

(i) \( e^x \) is continuous and differentiable for all \( x \).

(ii) \( e^x > 0 \) and \( e^{-x} = 1/e^x \) for all \( x \).

(iii) \( e^x \) is strictly increasing and concave up.

(iv) \( e^x \geq 1 + x \) for all \( x \).

(v) \( \lim_{x \to -\infty} e^x = \infty \) and \( \lim_{x \to \infty} e^x = 0 \).

**Proof**

(i) Since \( e^x \) solves the equation \( y' = y \), it must be differentiable. Moreover, every differentiable function is continuous (see, e.g. reference 1, p. 184).

(ii) If there is \( x \) such that \( e^x \leq 0 \), then, by the intermediate value theorem (see reference 1, p. 149), there must be \( x' \) such that \( \exp(x') = 0 \) (because \( \exp(0) = 1 > 0 \)). It follows that

\[
1 = e^0 = e^{x'+(-x')} = e^{x'} e^{-x'} \quad (\text{Fact 4}) = 0 \cdot e^{-x'} = 0 ,
\]

which is a contradiction. Hence, there is no \( x \) such that \( e^x \leq 0 \), i.e. \( e^x > 0 \) for all \( x \). It follows from (1) that \( e^{-x} = 1/e^x \).
(iii) Since, by definition and (ii),

\[(e^x)' = e^x > 0,\]

\(e^x\) is strictly increasing. Since, again by the definition and (ii),

\[(e^x)'' = ((e^x)')' = (e^x)' = e^x > 0,\]

\(e^x\) is concave up.

(iv) The tangent line to \(y = e^x\) at \(x = 0\) has the equation \(y = \exp(0) + \exp'(0)x = 1 + x\). Since, by (iii), \(e^x\) is concave up, the graph of \(e^x\) must be above the tangent line and the inequality follows.

(v) The first part follows directly from (iv). Indeed,

\[\lim_{x \to \infty} e^x \geq \lim_{x \to \infty} (1 + x) = \infty.\]

Thus, by (ii),

\[\lim_{x \to -\infty} e^x = \lim_{x \to -\infty} \frac{1}{\exp(-x)} = 0,\]

which proves the second part.

6. Conclusion and discussion

We have defined \(e^x\) in a relatively elementary way as the unique solution of the initial value problem

\[y'(x) = y(x), \quad y(0) = 1,\]

and showed how easily all properties of the exponential function follow. In particular, we have demonstrated the power of uniqueness. Indeed, the main idea behind most of the proofs was to guess a function, check its properties, and uniqueness guaranteed the rest.

Let us also mention that our approach is universal in some sense. It is relatively easy to see that the functions \(\cos x\) and \(\sin x\) can be defined in a very similar manner: as the real part and the imaginary part, respectively, of the unique solution of the initial value problem

\[y'(x) = iy(x), \quad y(0) = 1,\]

and clearly we do not have to stop here.

There are drawbacks to our approach. The most obvious one is the necessity to use Picard’s theorem—a deep statement behind most of the reasoning—which we used without a proof. We argue that this is not really a drawback, rather a standard trend in modern science. We see that trend even in mathematics. Very few people can completely or even partially understand Wiles’ proof of Fermat’s last theorem; see reference 10. Yet almost everybody understands the statement itself and its use. Moreover, mathematics nowadays is so broad and complex that the acceptance of a nontrivial statement without a proof is inevitable; confront reference 5.

In our opinion the more serious drawback is that with our approach the reader is literally shielded from the exposure to many beautiful mathematical concepts and ideas that would otherwise be needed to understand the exponential function in its full strength. We would be much happier if everybody had to study the exponential function from at least five different points of view, each one corresponding to a different definition and each one requiring a different
approach. However, one has to be realistic. For us, as mathematicians at heart, it may be hard to accept the existence of people whose sole purpose is not to study mathematics. Yet, such people exist. And for those people, these few pages may indeed constitute everything they need to know about $e^x$. Yes, they will not learn much mathematics. This hopefully means they will not be scared too much and possibly even like mathematics by suddenly seeing it as something understandable and not boring. And this is exactly what we wanted to achieve. After all, is it not every mathematician’s secret mission to convert as many people to be as devoted to mathematics as oneself?

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**References**


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\[
\begin{align*}
9^2 &= 81, \\
99^2 &= 9801, \\
999^2 &= 998001, \\
9999^2 &= 99980001, \\
99999^2 &= 999980001, \\
999999^2 &= 99999800001,
\end{align*}
\]

and so on.

10 Shahid Azam Lane, Makki Abad Avenue, Sirjan, Iran

Abbas Roohol Amini