POINTWISE UNIFORMLY ROTUND NORMS

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Abstract. It is shown that some properties of compact spaces $K$, such as carrying a strictly positive measure or being descriptive, are closely related to renormings of $C(K)$ or $C(K)^*$, respectively, by pointwise uniformly rotund norms.

Let $X$ be a Banach space. If $F$ is a closed, weak* dense subspace of $X^*$, then a norm $\|\cdot\|$ on $X$ is said to be $F$-uniformly rotund ($UR^F$) if $\lim_{n \to \infty} f(x_n - y_n) = 0$ for every $f \in F$ and every $x_n, y_n \in X$ such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n \to \infty} \|x_n + y_n\| = 2$. The norm is called pointwise uniformly rotund ($p$-$UR$) if it is $UR^F$ for some weak* dense $F \subset X^*$ (see [20], [19]). In particular, the norm on $X = Y^*$ is called weak* uniformly rotund if it is $UR^Y$ with the canonical embedding $Y \subset X^*$.

A measure $\mu$ on a compact space $K$ is said to be strictly positive if $\mu(U) > 0$ for every nonempty open set $U \subset K$. A compact space $K$ is called a uniform Eberlein compact if $K$ is homeomorphic to a weakly compact set in a Hilbert space [3]. A family $\mathcal{N}$ of subsets of a compact space $K$ is said to be a network if every open set in $K$ is a union of members of $\mathcal{N}$. A compact space $K$ is descriptive if there are closed sets $A_n \subset K$ and a network $\mathcal{N} = \bigcup_n \mathcal{N}_n$ such that every $\mathcal{N}_n$ consists of relatively open and pairwise disjoint sets in $A_n$ [18, Lemma 3.1]. A compact space $(K, \tau)$ is fragmentable, if there is a metric $\rho$ on $K$ such that for every $\varepsilon > 0$ and every nonempty subset $M \subset K$ there exists an $\varepsilon$-open set $\Omega \subset K$ such that $M \cap \Omega$ is nonempty and has $\rho$-diameter less than $\varepsilon$ ( [7], [10]). A Banach space $X$ is weakly compactly generated if there is a weakly compact set $K \subset X$ such that $X = \text{span} K$. For unexplained terms used in this paper we refer to [7] and [9]. Clearly, every $p$-UR norm is $UR^E$. $UR^E$ norms are used in fixed point theory; see e.g. [3]. It turned out that $p$-UR norms can be used in characterizing some properties of compact spaces as follows.

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Theorem 1. The space $C(K)$ of continuous functions on a compact space $K$ admits an equivalent pointwise uniformly rotund norm if and only if $K$ carries a strictly positive Radon probability.

Theorem 2. (a) For a compact space $K$, the space $C(K)^*$ admits a pointwise uniformly rotund (in general nondual) norm if and only if the space $L_1(\mu)$ is separable for every Radon probability on $K$.

(b) If $K$ is a descriptive compact space, then $C(K)^*$ admits an equivalent dual pointwise uniformly rotund norm.

(c) There is a nondescriptive (fragmentable) compact space $K$ such that $C(K)^*$ admits an equivalent dual pointwise uniformly rotund norm.

(d) If $K$ is a fragmentable compact space, then $C(K)^*$ admits an equivalent pointwise uniformly rotund norm. Consequently, the space $L_1(\mu)$ is separable for every Radon probability $\mu$ on a fragmentable compact $K$.

Theorem 3. Let $\mu$ be a finite measure. Then $L_1(\mu)$ admits an equivalent pointwise uniformly rotund norm if and only if $L_1(\mu)$ is separable.

Theorem 4. If a Banach space $X$ admits an equivalent pointwise uniformly rotund norm, then every weakly compact subset of $X$ is a uniform Eberlein compact.

Proof of Theorem 4. By Šmulyan’s type theorem [5, Theorem 2.6.7], if the norm $\|\cdot\|$ on a Banach space $X$ is UR$_F$, then the limit

$$\lim_{t \to 0} \frac{\|f + tg\|^* - \|f\|^*}{t}$$

exists for every $g \in X^*$, $\|g\|^* = 1$ and is uniform in $f \in F$, $\|f\|^* = 1$, where $\|\cdot\|^*$ is the dual norm to $\|\cdot\|$. In particular, the norm $\|\cdot\|^*$ is uniformly Gâteaux smooth on...
F. By [3], the dual unit ball $B_{F^*}$ is a uniform Eberlein compact in weak$^*$ topology of $F^*$. Hence, by [2], $F$ is a subspace of weakly compactly generated space $C(B_{F^*})$.

For a given weak$^*$ dense subspace $F \subseteq X^*$, let an operator $T : X \rightarrow F^*$ be given by $T = r \circ i$, where $i : X \rightarrow X^{**}$ is the canonical inclusion and $r : X^{**} \rightarrow F^*$ is the canonical restriction. The operator $T$ is one-to-one and $\sigma(X, X^*) - \sigma(F^*, F)$ continuous. Since $B_{F^*}$ is a uniform Eberlein compact in $\sigma(F^*, F)$ topology, $T(K)$ is a uniform Eberlein compact for every weakly compact set $K \subseteq X$. Hence $K$ is a uniform Eberlein compact, and the proof of Theorem 3 is finished.

Note that if $F$ admits a uniformly Gâteaux smooth norm, then $F^*$ admits a weak$^*$ uniformly rotund norm (see [5, Theorem 2.6.7]), and thus the norm $||.||$ on $X$ defined by
\[
||x||^2 = ||x||^2 + ||Tx||^2
\]
is an equivalent UR$^F$ norm.

**Proof of Theorem 1**. Let $\mu$ be a strictly positive Radon probability measure on $K$. Then the identity map $I : C(K) \rightarrow L_2(\mu)$ is one-to-one and with a dense range. Thus the norm $||.||$ defined on $C(K)$ by
\[
||f||^2 = ||f||^2 + ||f||^2_{L_2(\mu)}
\]
is an equivalent UR$^F$ norm, where $F = \overline{\text{span}} I(L_2(\mu)) \subseteq C(K)^*$.

Conversely, if $C(K)$ admits an equivalent UR$^F$ norm, then $F$ is a subspace of a weakly compactly generated space (see the proof of Theorem 3). Hence $\ell_1(\Gamma)$ is not a subspace of $F$ for any uncountable set $\Gamma$; see [9, Chapter 11]. By [17, Lemma 1.3], there is a Radon probability $\mu$ on $K$ such that $F \subseteq L_1(\mu) \subseteq C(K)^*$. Note that the measure $\mu$ is strictly positive as $F$ is weak$^*$ dense in $C(K)^*$. This concludes the proof of Theorem 1.

**Proof of Theorem 3**. If $L_1(\mu)$ is separable, then it admits an equivalent p-UR norm with the same proof as that of [3, Corollary 2.6.9]. Assume that $L_1(\mu)$ is nonseparable and admits an equivalent UR$^F$ norm. We claim that $F$ is norm separable. This means that $L_1(\mu)^*$ is weak$^*$ separable, which is a contradiction with [9, Theorem 11.3].

To prove our claim, let us identify $L_1(\mu)^* = L_\infty(\mu)$ with $C(\Omega)$, where $\Omega$ is a Stonian space for measure $\mu$ (see [4, Appendix B] for details). Since the measure $\mu$ is finite, the space $L_1(\mu)^*$ admits an equivalent weak$^*$ uniformly rotund norm. By Theorem 1 $\Omega$ carries a strictly positive probability measure. In particular, $\Omega$ has a property ccc, that is, every collection of pairwise disjoint open sets of $\Omega$ is countable. Thus we only need to prove the following fact, which is a version of [17, Theorem 4.5(a) and Proposition 4.7].

**Fact 5.** Let $\Omega$ be a compact space with property ccc and let $X \subseteq C(\Omega)$ be isomorphic to a subspace of a weakly compactly generated space. Then $X$ is separable.

**Proof.** By [7, Theorem 7.2], there exists a Markushevich basis of $X$, i.e., a biorthogonal system $\{x_\gamma, f_\gamma\}_{\gamma \in \Gamma} \subseteq X \times X^*$ such that $\bigcap_{\Gamma} \{x_\gamma ; \gamma \in \Gamma\} = X$ and $\{f_\gamma ; \gamma \in \Gamma\}$ separates points of $X$. We may and do assume that $\|x_\gamma\| = 3$. By [10], there exists a decomposition of $\Gamma = \bigcup_{n=1}^\infty \Gamma_n$ such that, for every $n \in \mathbb{N}$,
\[
\emptyset \neq \{x_\gamma ; \gamma \in \Gamma_n\}^{\infty}_{\gamma=1} (X^{**}, X^*) \setminus \{x_\gamma ; \gamma \in \Gamma_n\} \subseteq B_{X^{**}}.
\]
Proof of Theorem 2(b). Let $\|\cdot\|$ be the canonical dual norm on $C(K)^*$. Fix the family $\mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n$ given by the definition of descriptivity of $K$. Consider $\mathcal{R} \subseteq C(K)^*$ by the action $N(\mu) = \mu(N)$ for $N \in \mathcal{R}$ and $\mu \in C(K)^*$. Let $F = \text{span} \mathcal{R}$. We will show that there is an equivalent dual UR norm on $C(K)^*$.

We claim that $F$ is weak$^*$ dense in $C(K)^{**}$. Indeed, $\mu(G) = 0$ for all open $G \subseteq K$ whenever $\mu(N) = 0$ for all $N \in \mathcal{R}$, as $\mathcal{R}$ is a $\sigma$-isolated network consisting of relatively open pairwise disjoint sets.

Define a norm $\|\cdot\|$ on $C(K)^*$ in four steps, similarly as in [18, Proof of Theorem 3.3]. First, for every $n \in \mathbb{N}$, define a convex function $F_n$ on $C(K)^*$ by

$$F_n(\mu) = \sum_{N \in \mathcal{R}_n} |\mu|(N)^2.$$ 

The function $F_n$ is weak$^*$ lower semi-continuous on $C(A_n)^*$. Second, for every $n, m \in \mathbb{N}$, define a weak$^*$ lower semi-continuous seminorm $\|\cdot\|_{m,n}$ on $C(K)^*$ by

$$\|\mu\|_{m,n}^2 = \|\mu\|_1^2 + \sum_{m,n \in \mathbb{N}} 2^{-m-n}\|\mu\|_{m,n}^2.$$ 

Third, define an equivalent dual norm on $C(K)^*$ by

$$\|\mu\| = \|\mu\|_1^2 + \sum_{m,n \in \mathbb{N}} 2^{-m-n}\|\mu\|_{m,n}^2.$$ 

Claim 6.

$$\lim_{\omega \to \infty} (\mu_\omega - \nu_\omega)(N) = 0,$$

for all $n \in \mathbb{N}$, $N \in \mathcal{R}_n$ and all positive measures $\mu_\omega, \nu_\omega \in C(K)^*$, $\omega \in \mathbb{N}$, such that $\|\mu_\omega\|_1 \leq 1$, $\|\nu_\omega\|_1 \leq 1$, and

$$\lim_{\omega \to \infty} 2\|\mu_\omega\|_1^2 + 2\|\nu_\omega\|_1^2 - \|\mu_\omega + \nu_\omega\|_1^2 = 0.$$

Once the claim is proved, finally define a norm $\|\cdot\|$ by

$$\|\mu\| = \inf \left\{ \|\mu_1\|_1^2 + \|\mu_2\|_1^2 : \mu_i \in M(K), \mu_i \geq 0, \mu = \mu_1 - \mu_2 \right\}.$$ 

Using the compactness argument, it follows from the weak$^*$ lower semicontinuity of $\|\cdot\|_1$ that the infimum in $(5)$ is attained for every $\mu \in C(K)^*$ and that the norm $\|\cdot\|$ is an equivalent dual norm on $C(K)^*$. Thus $(5)$ holds whenever $\|\mu_\omega\| = 1 = \|\nu_\omega\|$ and $\lim_{\omega \to \infty} \|\mu_\omega + \nu_\omega\| = 2$. Hence the norm $\|\cdot\|$ is UR$^F$. 

Remark. After submission, we learned that Theorem 3 was proved by a different method in [6, Theorem 2.11].
Proof of Claim [3] Fix $n \in \mathbb{N}$ and $N \in \mathcal{N}_n$. From (3) and a convexity argument,

$$\lim_{\omega \to \infty} 2\|\mu_\omega\|_{m,n}^2 + 2\|\nu_\omega\|_{m,n}^2 - \|\mu_\omega + \nu_\omega\|_{m,n}^2 = 0,$$

for every $m \in \mathbb{N}$. From a compactness argument, for every $\omega, m \in \mathbb{N}$, there are positive measures $u_\omega^{m,n}, v_\omega^{m,n} \in C(A_n)^*$ such that

$$\|\mu_\omega\|_{m,n}^2 = \|\mu_\omega - u_\omega^{m,n}\|^2 + m^{-1}F_n(u_\omega^{m,n})^2$$

and

$$\|\nu_\omega\|_{m,n}^2 = \|\nu_\omega - v_\omega^{m,n}\|^2 + m^{-1}F_n(v_\omega^{m,n})^2.$$

Consequently,

$$F_n(u_\omega^{m,n}) \leq m\|\mu_\omega\|_{m,n} \leq m\|\mu_\omega\|_1 \leq m$$

and similarly $F_n(v_\omega^{m,n}) \leq m$. By passing to a subsequence, we may assume that

$$\lim_{\omega \to \infty} \|\mu_\omega\|_{m,n} = d_{m,n} = \lim_{\omega \to \infty} \|\nu_\omega\|_{m,n}.$$

The sequence $\{\|\mu\|_{m,n}\}_{m=1}^\infty$ is nonincreasing for every measure $\mu \in C(K)^*$. Thus there is $d_\omega = \lim_{m \to \infty} d_{m,n}$. Choose $\varepsilon > 0$ and let $m_0 \in \mathbb{N}$ be such that $d_{m_0,n} < d_\omega + \varepsilon$. We will estimate $|(\mu_\omega - \nu_\omega)(N)|$ by

$$\left|\left(\mu_\omega - u_\omega^{m_0,n}\right)(N)\right| + \left|\left(u_\omega^{m_0,n} - v_\omega^{m_0,n}\right)(N)\right| \leq \left|\left(\mu_\omega - v_\omega^{m_0,n}\right)(N)\right|.$$

By a convexity argument and (6), (7), (8),

$$\lim_{\omega \to \infty} 2F_n(u_\omega^{m_0,n})^2 + 2F_n(v_\omega^{m_0,n})^2 - F_n(u_\omega^{m_0,n} + v_\omega^{m_0,n})^2 = 0.$$

Since $u_\omega^{m_0,n}$ and $v_\omega^{m_0,n}$ are positive measures, by a convexity argument again

$$\lim_{\omega \to \infty} \left|\left(u_\omega^{m_0,n} - v_\omega^{m_0,n}\right)(N)\right| = 0.$$

In order to estimate $|(\mu_\omega - u_\omega^{m_0,n})(N)|$, consider a measure

$$u = \mu_\omega|_N + u_\omega^{m_0,n}|_{K \setminus N}$$

in the definition of $\|\mu_\omega\|_{m,n}$, where $\mu|_A$ means the restriction of $\mu$ on $A \subset K$. We get

$$\|\mu_\omega\|_{m,n}^2 \leq \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus N}\|_1^2 + m^{-1}F_n(\mu_\omega|_N + u_\omega^{m_0,n}|_{K \setminus N})^2$$

$$\leq \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus N}\|_1^2 + m^{-1}\left(F_n(\mu_\omega|_N) + F_n(u_\omega^{m_0,n}|_{K \setminus N})\right)^2$$

$$\leq \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus N}\|_1^2 + m^{-1}\left(\mu_\omega(N) + F_n(u_\omega^{m_0,n})\right)^2$$

$$\leq \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus N}\|_1^2 + m^{-1}\left(1 + m_0\right)^2.$$

Thus, for all $m \in \mathbb{N}$,

$$\limsup_{\omega \to \infty} \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus N}\|_1^2 \geq \lim_{\omega \to \infty} \|\mu_\omega\|_{m,n}^2 - m^{-1}(1 + m_0)^2,$$

$$\limsup_{\omega \to \infty} \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus N}\|_1^2 \geq d_{m,n}^2 - m^{-1}(1 + m_0)^2,$$

and

$$\limsup_{\omega \to \infty} \|(\mu_\omega - u_\omega^{m_0,n})|_{K \setminus N}\|_1^2 \geq d_n^2.$$

For all $\omega \in \mathbb{N}$ we have

$$\left|\left(\mu_\omega - u_\omega^{m_0,n}\right)(N)\right| \leq \left|\left(\mu_\omega - u_\omega^{m_0,n}\right)|_N\right|_1$$

$$= \left|\left(\mu_\omega - u_\omega^{m_0,n}\right)\right|_1 - \left|\left(\mu_\omega - u_\omega^{m_0,n}\right)|_{K \setminus N}\right|_1$$

$$\leq \|\mu_\omega|_{m_0,n} - \|\mu_\omega - u_\omega^{m_0,n}\|_{K \setminus N}\|_1.$$
Thus
\[
\lim \inf_{\omega \to \infty} (|\mu_{\omega} - \nu_{\omega}^{m_0,n}|(N)) \leq d_{m_0,n} - d_n \leq \varepsilon.
\]
The same estimate holds for \(||\nu_{\omega} - \nu_{\omega}^{m_0,n}|(N)|\). The proof of Claim 8 is complete.

**Proof of Theorem 2.**} First, we prove the following claim, which is a version of [11, Theorem 7.1].

**Claim 7.** Suppose that on a tree \(T\) there is an increasing function \(g: T \to \mathbb{R}\) which is constant on no strictly increasing sequence in \(T\). Then there is an equivalent dual \(p\)-UR norm on \(C_0(T)^*\).

**Proof of Claim 7.** The space \(C_0(T)^*\) can be identified with \(\ell_1(T)\) with the canonical dual norm \(\|\mu\|_1 = \sum_{t \in T} |\mu(t)|\). Let us define \(T^+\) as the set of successors and \(T_0\) as the set of all \(t \in T^+\) such that \(g(t) > g(t^-)\). We may modify the function \(g\) so that it takes rational values at all points of \(T_0\).

We will show that there is an equivalent dual UR\(F\) norm where
\[
F = \text{span}\left\{ \left\{ s \right\}; s \in T^+ \right\} \cup \left\{ \left[ s, \infty \right); s \in T_0 \right\} \cup \left\{ T \right\} \subset \ell_\infty(T).
\]
We claim that \(F\) is weak\(^*\)-dense in \(C_0(T)^{**}\). To prove it, let \(\mu \in C(T)^*\) be such that \(\mu(f) = 0\) for all \(f \in F\). We want to show that \(\mu(t) = 0\) for all \(t \in T\). Choose \(t \in T\) and put \(A(t) = \{ u; u \in (t, \infty), g(u) = g(t) \}\) and \(B(t) = \min\{ u \in (t, \infty); g(u) > g(t) \}\). We have
\[
(t, \infty) = \bigcup_{u \in A(t)} \{ u \} \cup \bigcup_{u \in B(t)} \left[ u, \infty \right).
\]
The union above is a union of disjoint open sets and \(|\mu|\) is nonzero at most on countably many of them. Hence \(\mu((t, \infty)) = 0\). Thus \(\mu([t, \infty)) = 0\) for all \(t \in T^+\). Since
\[
(0, t] = T \setminus \bigcup_{s \leq t} \left( \bigcup_{r \in s^- \setminus \{0, t\}} [r, \infty) \right),
\]
we have that \(\mu((0, t]) = 0\) for all \(t \in T\). Every limit element \(t \in T\) is a limit of a sequence (of elements of \(T_0\)), thus \(\mu((0, t]) = 0\) for all \(t \in T\). Hence \(\mu(\{ t \}) = 0\) for all \(t \in T\).

For every \(q \in \mathbb{Q}\), the wedges \([s, \infty)\) with \(s \in T_0\) and \(g(s) = q\) are disjoint, so we can define an equivalent dual norm on \(C(T)^*\) by
\[
\|\mu\|^2 = \|\mu\|^2 + \sum_{s \in T^+} \|\mu(I_s)\|^2 + \sum_{q \in \mathbb{Q}} c_q \left( \sum_{s \in T_0 \cap q^{-1}(q)} \|\mu|_{[s, \infty)}\|^2 \right),
\]
where \(c_q\) are some positive constants.

Let \(\mu_n, \nu_n \in C_0(T)^*\) be positive elements such that \(\|\mu_n\| \leq 1, \|\nu_n\| \leq 1\) and
\[
\lim_{n \to \infty} 2\|\mu_n\|^2 + 2\|\nu_n\|^2 - \|\mu_n + \nu_n\|^2 = 0.
\]
A standard convexity argument shows that
\[
\lim_{n \to \infty} (\mu_n - \nu_n)(T) = 0, \lim_{n \to \infty} (\mu_n - \nu_n)(s) = 0,
\]
for all \(s \in T^+\), and
\[
\lim_{n \to \infty} (\mu_n - \nu_n)([s, \infty)) = 0,
\]
for any \(s \in T_0\). Thus the norm \(\|\|\) defined by (5) is UR\(F\). This concludes the proof of Claim 7.
Now, let $\Lambda$ be a tree defined in [11, Section 10] and let $K$ be its Alexandroff compactification. Then $C(K)^*$ admits an equivalent dual p-UR norm by Claim 4. The space $C(K)^*$ does not admit any equivalent dual locally uniformly rotund norm, since $C(K)$ does not admit an equivalent Fréchet smooth norm [11, Corollary 10.9]. Thus $K$ is not a descriptive compact space by [13, Corollary 4.9]. The proof of Theorem 2(c) is complete.

**Proof of Theorem 2(d).** Let $K$ be a fragmentable compact. By [2] Theorem 5.1.9 and Proof of Theorem 5.1.12(iii), there is a family $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ of subsets of $K$ such that

1. $\mathcal{U}$ is a separating family, i.e. if $x \neq y \in K$, then there is $U \in \mathcal{U}$ such that $\# U \cap \{x, y\} = 1$;
2. $\mathcal{U}$ is a network;
3. for every $n \in \mathbb{N}$, $\mathcal{U}_n$ is an open partitioning, i.e. $\mathcal{U}_n = \{U_\xi; \xi < \xi_n\}$ is well ordered such that $U_\xi$ is contained and is relatively open in $K \setminus (\bigcup_{\eta < \xi} U_\eta)$ for every $\xi < \xi_n$ and $K = \bigcup_{\xi < \xi_n} U_\xi$;
4. for every $U \in \mathcal{U}_{n+1}$ there is $V \subset \mathcal{U}_n$ such that $\overline{U} \subset V$.

As $\mathcal{U}_n$ is an open partitioning, it follows that

$$\sum_{U \in \mathcal{U}_n} \mu(U) = \mu(K).$$

Define equivalent norms on $C(K)^*$

$$\|\mu\|_2^2 = |\mu|^2(K) + \sum_{n=1}^{\infty} 2^{-n} \sum_{U \in \mathcal{U}_n} |\mu|^2(U)$$

and

$$\|\mu\|^2 = \inf \{\|\mu_1\|^2_+ + \|\mu_2\|^2_-; \mu_i \in C(K)^*, \mu_i \geq 0, \mu = \mu_1 - \mu_2\}.$$

From a definition of a norm $\|\cdot\|_+$ it follows that $\|\mu\|^2 = \|\mu^+\|^2_+ + \|\mu^-\|^2_-$. Let $F = \text{span}\{U; U \in \mathcal{U}\} \subset C(K)^{**}$. We will show that the norm $\|\cdot\|$ is UR$^F$. Note that $F \subset C(K)^{**}$ is weak$^*$ dense. Indeed, assume $\mu(U) = 0$ for all $U \in \mathcal{U}$ and let $G \subset K$ be an open set. Since $\mathcal{U}$ is a network, we have $G = \bigcup_{n=1}^{\infty} \bigcup_{U \in \mathcal{U}_n} U$, where for every $n \in \mathbb{N}$, $\mathcal{U}_n$ is a subfamily of $\mathcal{U}_m$. Moreover, by condition (3), we may assume that $\mathcal{U}_m \cap \mathcal{U}_m' = \emptyset$ for $m \neq n$. Thus

$$\mu(G) = \mu\left(\bigcup_{n=1}^{\infty} \bigcup_{U \in \mathcal{U}_n} U\right) = \sum_{n=1}^{\infty} \mu\left(\bigcup_{U \in \mathcal{U}_n} U\right) = \sum_{n=1}^{\infty} \sum_{U \in \mathcal{U}_n} \mu(U) = 0,$$

where the third equality hold as $\mathcal{U}_n$’s are relatively open partitioning. Thus, by a convexity argument, the norm $\|\cdot\|$ is UR$^F$.

The proof of Theorem 2(d) is complete.

**References**


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