ADDITIONAL EXERCISES: TUESDAY

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(1) Show $|tζ(1 + it)| ≥ 1$ for all $t ∈ \mathbb{R}$.
(2) Is $\text{Re}(ζ(1 + it)) > 0$ for $t ≠ 0$?
(3) Explain the “paradox”: Define

$$ζ(s) = \prod_p \frac{1}{1 - p^{-s}} \quad \text{for } σ > 1,$$
$$F(s) = \prod_p \frac{1}{1 + p^{-s}} \quad \text{for } σ > 1.$$  

Since these are absolutely convergent, we can multiply together to show

$$ζ(s)F(s) = \prod_p \frac{1}{1 - p^{-2s}} = ζ(2s).$$

We have two meromorphic function that agree for $σ > 1$, so their analytic continuations agree

$$ζ(s)F(s) = ζ(2s).$$

What happens on the critical line?

$$ζ(\frac{1}{2} + it)F(\frac{1}{2} + it) = ζ(1 + 2it) ≠ 0.$$  

Thus, there are no zeros on the critical line. Explain.