Squeeze, Pinch or Sandwich Theorem
This article has no author or citation because it is found in most Calculus books.

Let \( f, g, h \) be three functions defined on an interval \( [a, b] \). If \( a < c < b \) and \( f(x) \leq g(x) \leq h(x) \) for all \( a \leq x \leq b \) and \( \lim_{x \to c} f(x) = L \), and \( \lim_{x \to c} h(x) = L \), then

\[
\lim_{x \to c} g(x) = L
\]

Proof. (For a diagram of the theorem see the current Calculus book). It is required to show that for any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that

\[
0 < |x - c| < \delta \Rightarrow |g(x) - L| < \varepsilon
\]

So let \( \varepsilon \) be an arbitrary positive number. We are given \( \lim_{x \to c} f(x) = L \), and \( \lim_{x \to c} h(x) = L \). Therefore we are given

(i) \quad \text{there is } \delta_1 > 0 \text{ such that } 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \text{ and}

(ii) \quad \text{there is } \delta_2 > 0 \text{ such that } 0 < |x - c| < \delta \Rightarrow |h(x) - L| < \varepsilon \text{ and}

We are also given

(iii) \quad f(x) \leq g(x) \leq h(x) \text{ for all } a \leq x \leq b.

Now we can take \( \delta = \min\{\delta_1, \delta_2\} \), and verify (*):

\[ L - \varepsilon < f(x) < L + \varepsilon \text{ and } L - \varepsilon < h(x) < L + \varepsilon. \text{ Thus}
\]

\[ L - \varepsilon \leq f(x) \leq g(x) \leq h(x) \leq L + \varepsilon \text{ (using parts of (i), (ii) and (iii)) which is the same as saying}
\]

\[ |g(x) - L| < \varepsilon
\]

This completes the proof.

Exercise: Application of the Squeeze Theorem.

(a) Explain using graphs why \( \lim_{x \to 0} \sin \frac{1}{x} \) does not exist [Hint: First graph values of \( x \) of the form \( \frac{1}{\pi/2}, \frac{1}{3\pi/2}, \frac{1}{5\pi/2}, \cdots, \frac{1}{(2n+1)\pi/2} \), and then graph values of \( x \) of the form \( \frac{1}{5\pi/2}, \frac{1}{7\pi/2}, \cdots, \frac{1}{(2n+1)\pi/2} \cdots \)]

(b) Explain why \( \lim_{x \to 0} x \sin \frac{1}{x} \) cannot be found using the product rule for limits. [Hint: The product rule for limits says

\[
\lim_{x \to a} (f(x) \cdot g(x)) = (\lim_{x \to a} f(x)) \cdot (\lim_{x \to a} g(x))
\]

provided both limits on the right hand side exist

(c) Find \( \lim_{x \to 0} x \sin \frac{1}{x} \).

\[ \Box \]