Equilateral sets in $\ell^d_p$

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Abstract If $X$ is a Minkowski space, i.e. a finite dimensional real normed space, then $S \subset X$ is an equilateral set if all pairs of points of $S$ determine the same distance with respect to the norm. Kusner conjectured that $e(\ell^d_p) = d + 1$ for $1 < p < \infty$ and $e(\ell^d_1) = 2d$ [6]. Using a technique combining linear algebra and approximation theory, we prove that for all $1 < p < \infty$, there exists a constant $C_p > 0$ such that $e(\ell^d_p) \leq C_p d^{1 + 2/(p-1)}$.

Keywords Minkowski spaces, equilateral sets, Kusner’s conjecture, linear algebra method, approximation by polynomials, $k$-distance sets.

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1 Introduction

Let $(X, \| \cdot \|)$ be a finite-dimensional real normed space, also called a Minkowski space. Let $\Delta(S) := \{\|s - t\| : s, t \in S, s \neq t\}$. For $k \geq 1$, we say $S$ is a $k$-distance set if $|\Delta(S)| \leq k$. Let $e_k(X)$ be the maximum possible cardinality of a $k$-distance set in $X$. This maximum exists: $e_k(X) \leq (k + 1)^{\dim(X)}$ [11]. We call a 1-distance set an equilateral set and write $e(X) := e_1(X)$.

Petty showed that $e(X) \leq 2^{\dim(X)}$ and that equality is obtained if and only if $X$ is affinely isometric to $\ell^d_\infty$ [10], Brass showed $e(X) = \Omega((\log d / \log \log d)^{1/3})$ [4]. Subsequently, Swanepoel and Villa showed that $e(X) \geq \exp(c \sqrt{d})$ for some $c > 0$ [13]. It is an open question whether $e(X) \geq \dim(X) + 1$ [5]. This has been proved for $\dim(X) = 3, 4$ [10,8].

We will consider the case where $X = \ell^d_p$, i.e. $\mathbb{R}^d$ in the $L^p$ norm. Kusner conjectured that $e(\ell^d_p) = d + 1$ for $1 < p < \infty$ and $e(\ell^d_1) = 2d$ [6]. It is well known that
$e(\ell_p^d) = d + 1$. We have $e(\ell_p^d) \geq d + 1$ as the standard basis vectors together with an appropriate scalar multiple of $(1, 1, \ldots, 1)$ form an equilateral set in $\ell_p^d$. Using a technique combining linear algebra and approximation theory, we show that

**Theorem 1** For all $1 < p < \infty$ there is a constant $C_p > 0$ such that

$$e(\ell_p^d) \leq C_p d^{1 + 2/(p-1)}.$$

This was later improved by Alon and Pudlák to $e(\ell_p^d) \leq C'_p d^{1 + 3/(2p-1)}$ for $1 \leq p < \infty$ and $e(\ell_p^d) \leq C''_p d \log d$ for $p \geq 1$, an odd integer [1]. Here $C'_p$ and $C''_p$ are positive constants. Galvin noted that $e(\ell_p^d) \leq 1 + (p-1)d$ for $p$ an even integer (personal communication, 1999). Swanepoel proved $e(\ell_p^d) \leq (2\lfloor p/4 \rfloor - 1)d + 1$ for $p$ an even integer [12]. In particular, $e(\ell_2^d) = d + 1$. Swanepoel also proved (in particular) that for every $1 \leq p < 2$, $e(\ell_p^d) > d + 1$ for $d$ large enough [12].

We will prove Theorem 1 and then make some conjectures concerning $e_k(X)$.

### 2 Proof of Theorem 1

Before proving Theorem 1, we require the following three easy or well-known results. Let $\text{Mat}_m$ be the set of $m \times m$ real matrices. For $M \in \text{Mat}_m$, let $\|M\|_\infty := \max_{1 \leq i,j \leq m} |M_{ij}|$. Let $I_m$ denote the $m \times m$ identity matrix.

**Lemma 1** [2] Let $V$ be the vector space of real-valued functions on a set $X$. Let $\{f_1, \ldots, f_m\} \subset V$. Let $\{a_1, \ldots, a_m\} \subset X$. Let $M \in \text{Mat}_m$ be the matrix with $M_{ij} = f_i(a_j)$. If $M$ is invertible then the $f_i$ are linearly independent.

**Lemma 2** [14] If $M \in \text{Mat}_m$ and $\|M - I_m\|_\infty < 1/m$, then $M$ is invertible.

**Lemma 3** Let $1 < p < \infty$. There is a sequence of polynomials $\{q_i(x)\}_{i \geq \lceil p \rceil}$ with $	ext{deg}(q_i) \leq i$, such that $\|x^p - q_i(x)\| \leq B_p/i^p$, for all $|x| \leq 1$ where $B_p > 0$ is a constant.

This last lemma is an application of the following result of Jackson [7]. For the statement given here see pg. 57 of [9].

**Theorem 2** Let $k \geq 1$. Suppose $f \in C^k([-1, 1])$ and $f^{(k)} \in \text{Lip}_M \alpha$. For every $l > k$ there is a polynomial $q_l$ of degree at most $l$ such that $\|f - q_l\|_\infty \leq D(k, \alpha, l)c^{k + 1}M/l^{k + \alpha}$ where $D(k, \alpha, l) = l^{k + \alpha}/((l)_k(l-k)!)^{\alpha}$ and $c = 1 + \pi^2/2$.

Here $\text{Lip}_M \alpha$ is the class of functions $f(x)$ on $[-1, 1]$ such that $|f(x) - f(y)| \leq M|x - y|^\alpha$ for all $x, y \in [-1, 1]$ and $\|f(x)\|_\infty = \max_{x \in [-1, 1]} f(x)$. As is common notation, $(l)_k$ is the falling factorial, $(l)_k := l(l-1)\cdots(l-k+1)$.

To obtain Lemma 3 we first note that each factor of $D(k, \alpha, l) = \prod_{i=1}^{k-1}(1 + i/(l - i))(1 + k/(l - k))^{\alpha}$ is decreasing with $l$. We thus obtain an upper bound for this quantity by setting $l = k + 1$, namely $D(k, \alpha, l) \leq (k + 1)^{k + \alpha}/(k + 1)!$. If $f(x) = |x|^p$ set $k = \lceil p \rceil - 1$ and $\alpha = p - k \in (0, 1)$. Then $f \in C^k([-1, 1])$ and $f^{(k)}(x) = \text{sgn}^k(x)(p)_{k}|x|^p \in \text{Lip}_M \alpha$ where $M = (p)_k$. Thus we obtain Lemma 3 with $B_p = (\lceil p \rceil^p(1 + \pi^2/2)/p)! |(p)_{\lceil p \rceil - 1}|/|p|!$. It is straightforward to verify that $B_p \geq |p|^p$ which fact we require in the proof of Theorem 1.
Proof (Theorem 1.) Fix \( p \in (1, \infty) \). Let \( S = \{a^1, a^2, \ldots, a^m\} \subset \ell_p^d \) be an equilateral set of maximum size, scaled so that \( ||a^i - a^j||_p = 1 \) for all \( i \neq j \). We define the following functions from \( \mathbb{R}^n \) to \( \mathbb{R} \), \( f_i(x) = 1 - ||x - a^i||_p^p = 1 - \sum_{j=1}^d |x_j - a^i_j|^p, \) \( i = 1, \ldots, m \) and \( g_i(x) = 1 - \sum_{j=1}^d q_j(x_j - a^j). \) \( i = 1, \ldots, m, \) where \( q_j \) is the polynomial from Lemma 3 approximating \( |x|^p \) to within \( B_p/|p| \) on \([-1, 1]\) and where \( l \) is smallest integer such that \( dB_p/|p| < 1/m \). Note that \( l > (dB_pm)^{1/p} \geq B_{l/p}^{1/p} \geq [p] \) as required by Lemma 3. Since \( B_p \geq 1, \) \( (dB_pm)^{1/p} \geq 1 \) and \( l \leq (dB_pm)^{1/p} + 1 \leq 2(dB_pm)^{1/p}. \)

Let \( M = [g_i(a^j)] \in \text{Mat}_m. \) By the assumption on \( S \) we have \( [f_i(a^j)] = I_m. \) Let \( A = M - I_m = [g_i(a^j) - f_i(a^j)]. \) We have \( |A_{ij}| \leq \sum_{j=1}^d |\zeta_{ij}| \) where \( \zeta_{ij} := |a^j_i - a^i_j|^p - q_j(a^j_i - a^i_j). \) Since \( |a^j_i - a^i_j| \leq ||a^j - a^i||_p \leq 1, \) we have \( |\zeta_{ij}| \leq B_p/|p| \) by Lemma 3. Thus \( \|M - I_m\|_\infty \leq dB_p/|p| < 1/m. \) This implies \( M \) is invertible by Lemma 2, and thus the \( g_i \) are linearly independent by Lemma 1.

The \( g_i \) are elements of \( W, \) the subspace of \( \mathbb{R}[x_1, \ldots, x_d] \) spanned by \( \{1, \sum_{j=1}^d x_j^d, \ldots, x_d^{-1}\}. \) Thus we have \( m \leq \dim(W) = 2 + d(l - 1) \leq dl, \) if \( d \geq 2. \) Since \( l \leq 2(dB_pm)^{1/p}, \) \( m \leq 2d(dB_pm)^{1/p}, \) or \( e(\ell_p^d) = m < 2^{p/(p-1)}B_{1/l}^{1/(p-1)d(p+1)/(p-1)}. \)

Notes: (i) The constant \( 2^{p/(p-1)}B_{1/l}^{1/(p-1)} \) grows without bound as \( p \) approaches 1 and is \( O(p) \) as \( p \) goes to infinity. (ii) As Galvin noted (personal communication, 1999), if \( p \) is an even integer the functions \( f_i(x) \) are polynomials of degree \( p, \) and together with the identity function, 1, form a linearly independent set in \( W. \) Thus \( e(\ell_p^d) \leq 1 + d(p-1). \)

3 k-distance sets

For \( d \geq 1, \) \( e(X) \) is upper semi-continuous on the Banach-Mazur compactum of normed spaces of dimension \( d. \) For a proof: suppose \( N_n \) is a sequence of norms converging to the norm \( N \) in the Banach-Mazur distance and \( S_n = \{a_{n,1}, \ldots, a_{n,e}\} \) is an equilateral set of size \( e \) in \( N_n \) with \( \Delta(S_n) = \{1\}, \) then there is an equilateral set \( S \) of size \( e \) in \( N. \) Indeed, we may assume by translating each \( S_n \) that \( \cup S_n \) lies in a compact set. By passing to convergent sub-sequences we get \( a_{n,i} \to a_i \) for all \( 1 \leq i \leq e. \) \( S = \{a_1, \ldots, a_e\} \) is equilateral in \( N. \) A particular instance: \( e(X) \leq d + 1 \) for \( X \) sufficiently close to \( \ell_2^d. \)

It seems plausible that

Conjecture 1 For all \( d, k \geq 1, \) \( e_k(X) \) is upper semi-continuous.

Trying the same approach used for \( e(X) \) leads to a problem. Setting \( \Delta(S_n) = \{1 = d_{n,1} > d_{n,2} > \cdots > d_{n,k}\} \) we can pass to sub-sequences to ensure \( ||a_{n,i} - a_{n,j}||_n = d_{f(i,j)} \) where \( f \) is a fixed function, \( d_{n,i} \to d_i \) and \( ||a_i - a_j|| = d_{f(i,j)}, \) but we may have \( d_k = 0 \) and \( |S| < e. \) This state of affairs would be rectified by proving the following conjecture.

Conjecture 2 For all \( d, k \geq 1, \) there is a universal constant \( c_{d,k} > 0 \) so that for any normed space \( X \) of dimension \( d \) there is a maximum size \( k \)-distance set \( S \subset X \) with maximum distance 1 and minimum distance greater than or equal to \( c_{d,k}. \)
We’d like to prove a $k$-distance analogue of Theorem 1. It is known that $e_k(e_p^d) \leq e_k(e_{\ell_p}^d) \leq {d+k \choose k}$ [3]. Mimicking the proof of the lower bound in the previous statement, it is trivial to show that the $d \choose k$ 0-1 vectors of length $d$ with exactly $k$ 1’s form a $k$-distance set in $\ell_p^d$, so that $e_k(e_{\ell_p}^d) \geq {d \choose k}$.

Conjecture 3 For all $k \geq 1$ there exists $p(k)$ so that for all $p > p(k)$ there exists a constant $C_{p,k} > 0$ so that $e_k(e_{\ell_p}^d) \leq C_{p,k}d^k$.

The natural attempt at a proof of this conjecture would be to define for $S = \{a^1, \ldots, a^m\} \subset \ell_p^d$ the functions $f_i(x) = \prod_{s=1}^k \left( d_{p,s}^d - \sum_{t=1}^d |x_t - a^s_t|^p \right)$ and the polynomial approximations $g_i(x) = \prod_{s=1}^k \left( d_{p,s}^d - \sum_{t=1}^d g_{s,t}(x_t - a^s_t) \right)$ where $\Delta(S) = \{1 = d_1 > d_2 > \cdots > d_k\}$. However when trying to prove $M = [g_i(a^s)]$ is invertible, the sufficient condition $\|M - [f_i(a^s)]\|_\infty = \|M - d_{p,1}^d \cdots d_{p,k}^d I_m\|_\infty < d_{p,1}^d \cdots d_{p,k}^d / m$ involves an estimate that could require an arbitrarily high degree of approximation $l$, as $d_k$ could be arbitrarily small. One needs lower bounds on $d_k$ in order to make this approach work.

The following is a seemingly reasonable conjecture.

Conjecture 4 For all $p \geq 1$ and all $k \geq 1$ there exists a constant $c_{p,k} > 0$ independent of $d$ so that there exists a $k$-distance set $S \subset \ell_p^d$ with $|S| = e_k(e_{\ell_p}^d)$ and with $\Delta(S) = \{1 = d_1 > d_2 > \cdots > d_k \geq c_{p,k}\}$.

Assuming this conjecture, we can prove

Conjecture 5 For all $k \geq 1$ and for all $p > k$ there exists a constant $C_{p,k} > 0$ so that $e_k(e_{\ell_p}^d) \leq C_{p,k}d^k((k^2+k)/(p-k))$.

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References