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A note on the maximal coefficients of squares of Newman polynomials

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Abstract

In a recent paper [G. Yu, An upper bound for $B_2[g]$ sets, J. Number Theory 122 (1) (2007) 211–220] Gang Yu stated the following conjecture: Let $\{p_i\}_{i=1}^{\infty}$ be an arbitrary sequence of polynomials with increasing degrees and all coefficients in $\{0, 1\}$. If we denote by $(\#p_i)$ the number of non-zero coefficients of p_i, and let $\mathfrak{M}(p_i^2)$ be the maximal coefficient of p_i^2, then

$$Q := \liminf_{i\to\infty} \frac{\deg(p_i)\mathfrak{M}(p_i^2)}{(\#p_i)^2} \geqslant 1, \tag{$*$}$$

as long as $(\#p_i) = o(\deg p_i)$, as $i \to \infty$. We give an explicit example that shows why this last condition is necessary, and we investigate some open questions it suggests.

1. Introduction

The aim of this short note is to study extremal properties of Newman polynomials. A Newman polynomial (see [9]) is a polynomial with coefficients in the set $\{0, 1\}$ (see also [2–4]). Over the last several decades, properties of polynomials with restricted coefficients have been investigated extensively and a wide variety of results have been proved concerning Newman polynomials, as well as the related Littlewood polynomials and cyclotomic polynomials (see [1,2,5,6,8,10,11]).

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But in spite of continuous interest in these topics, and wide applicability of results concerning them (in particular, see [12]), interesting questions dealing with the behavior of coefficients of *powers* of Newman and Littlewood polynomials seem to have been neglected. Even the simplest case—the squares of Newman polynomials, as covered by the conjecture of Yu [12], suggests several intriguing unsolved questions.

First, let us explain why the conjecture given in the abstract—even without the last restriction $(\#p_i) = o(\deg p_i)$—seems "plausible" to begin with. If one wishes to maximize coefficients of squares of Newman polynomials, it seems natural to begin by looking at examples with "many" coefficients equal to 1. In the extreme case where all the coefficients are 1, i.e.

$$f_i(x) := \sum_{m=0}^{i} x^m = 1 + x + x^2 + \cdots + x^{i-1} + x^i,$$

we have $\deg(f_i) = i$, and $(\#f_i)$, the number of non-zero coefficients of $f_i(x)$, is equal to $i + 1$. Also, in this case the square of $f_i(x)$ has the form

$$\begin{aligned} f_i(x)^2 &= \sum_{m=0}^{i}(m+1)x^m + \sum_{m=i+1}^{2i}(2i+1-m)x^m \\ &= 1 + 2x + 3x^2 + \cdots + (i+1)x^i + \cdots + 3x^{2i-2} + 2x^{2i-1} + x^{2i}, \end{aligned}$$

and hence $\mathfrak{M}(f_i^2) = i + 1$. It follows that in this particular case

$$Q = \liminf_{i\to\infty} \frac{\deg(f_i)\mathfrak{M}(f_i^2)}{(\#f_i)^2} = \lim_{i\to\infty} \frac{i(i+1)}{(i+1)^2} = 1. \tag{1}$$

What Yu's conjecture asserts is that no infinite family of polynomials satisfying the condition $(\#p_i) = o(\deg p_i)$ can give a better ratio Q. In fact, in the view of the above "extreme" example (1), it does not seem too unreasonable to expect that the upper bound 1 could work for all families of polynomials, not just those satisfying the condition. However, in what follows we show why this is not the case.

2. Main theorem

Our main goal is to prove the following theorem.

Theorem 1. *For $n = 4k$, define a family $\mathfrak{G} = \{g_n\}$ of Newman polynomials of degree n via the identity*

$$xg_n(x) := \sum_{m=1}^{n/4} x^m + \sum_{m=n/2}^{n} x^{m+1}. \tag{2}$$

Then

$$\lim_{k\to\infty} \frac{\deg(g_{4k})\mathfrak{M}(g_{4k}^2)}{(\#g_{4k})^2} = \frac{8}{9}. \tag{3}$$

Proof. For any $k \in \mathbb{N}$, let the polynomials $g_{4k}(x)$ be defined as in (2). Then clearly $\deg(g_{4k}) = 4k$ and $(\#g_{4k}) = 3k$. Moreover, the square $g_{4k}^2(x)$ of the polynomial $g_{4k}(x)$ (for simplicity multiplied by x) can be expanded as

$$x g_{4k}^2(x) = \sum_{m=1}^{k} m x^m + \sum_{m=k+1}^{2k} (2k-m)x^m + \sum_{m=2k+1}^{3k} 2(m-2k)x^m + \sum_{m=3k+1}^{4k} (2k)x^m + \sum_{m=4k+1}^{5k} (6k+2-m)x^m + \sum_{m=5k+1}^{6k} (m-4k)x^m + \sum_{m=6k+1}^{8k+1} (8k+2-m)x^m,$$

which implies that $\mathfrak{M}(g_{4k}^2) = 2k+1$, as one can readily verify. Therefore, for polynomials in $\mathfrak{G}$, we have

$$\lim_{k\to\infty} \frac{\deg(g_{4k})\mathfrak{M}(g_{4k}^2)}{(\#g_{4k})^2} = \lim_{k\to\infty} \frac{(4k)(2k+1)}{(3k)^2} = \frac{8}{9}. \qquad \square$$

Remark. Clearly, this implies that the restriction in Yu's conjecture is necessary.

3. Open questions

The following related questions seem to have useful applications in the theory:

- Without the restriction $(\#p_i) = o(\deg p_i)$, is it true that $8/9$ is the lowest possible value the quantity Q can ever attain? (We suspect that this is true.)
- Then, is it also true that $Q = 8/9$ occurs only for families of polynomials for which

$$\frac{(\#p_i)}{\deg(p_i)} \to \frac{3}{4}?$$

- How does the function

$$Q = \liminf_{i\to\infty} \frac{\deg(p_i)\mathfrak{M}(p_i^2)}{(\#p_i)^2}$$

  behave if we fix the quantity $\frac{(\#p_i)}{\deg(p_i)} = C$, for some constant $0 < C \leqslant 1$?
- In applications concerning $B_2[g]$ sets (as defined in [12]), what is the structure of coefficients of Newman polynomials that maximize Q in situations when

$$(\#p_i) \sim \sqrt{\deg(p_i)}?$$

- What can be said about extremal properties of coefficients of higher powers of Newman polynomials?

Remark. Another, perhaps even more interesting, set of related questions concerns the behavior of Littlewood polynomials. A Littlewood polynomial is a polynomial with coefficients in the set $\{+1, -1\}$ (see [7]). Analogs of the above questions seem to be even more delicate for squares and higher powers of Littlewood polynomials.

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