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[Bauer, Friedrich L.](#)**Why Legendre made a wrong guess about $\pi(x)$, and how Laguerre's continued fraction for the logarithmic integral improved it.***Math. Intelligencer* **25** (2003), *no. 3*, 7–11.[11N05 \(01A05 11-03\)](#)

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Prime numbers have been studied for over 25 centuries now, and the theory of their distribution is still the most fascinating topic in mathematics. The counting function $\pi(x) = \sum_{p \leq x} 1$ has been prominent as the subject of research ever since the end of the 18th century. The short paper under review discusses the early history of the subject centered around the classical formulas of Gauss and Legendre. It explains why Chebyshev's formula $x/(\log x - 1)$ is better than Legendre's enigmatic $\text{le}(x) := x/(\log x - 1.08366 \dots)$, which Abel called the "most beautiful of mathematics". In fact, this follows from the expression $x/(\log x - 1) \sim x/\log x + x/(\log x)^2$, which has an additional term in common with the expansion obtained by integration by parts of Gauss' logarithmic integral $\text{li}(x) := \int_2^x dt/\log t$, that can be proved to be more accurate than either of the above two approximations.

The author (p. 9) uses Laguerre's (1885) continued fraction expansion for $\text{li}(x)$,

$$\text{li}(x) = \frac{x}{\log x - 1 - \frac{1}{\log x - 3 - \frac{4}{\log x - 5 \dots}}},$$

in order to obtain the following approximation to $\pi(x)$:

$$\text{la}(x) := \frac{x}{\log x - 1 - \frac{1}{\log x - 3}}.$$

Several tables comparing the values of $\pi(x)$ with $x/\log x$, $\text{li}(x)$, $\text{la}(x)$ and $x(\log x - 1)$ are presented; some of them are based on recent computations of Deléglise and Gourdon.

In just four pages the author manages to condense a lot of interesting information. However, Riemann is mentioned in the paper only once, in passing, in spite of the fact that his ideas concerning the way to approximate $\pi(x)$ (his well-known $J(x)$ function) were not only the most accurate, but also the deepest, and eventually led to the first proof of the prime number theorem, namely $\pi(x) \sim \text{li}(x)$. One other intriguing question that is not discussed here is why Legendre used $A = 1.08366 \dots$ as the constant in $\text{le}(x)$. A detailed analysis of the original method he employed (related to his improvement of the Eratosthenes sieve) reveals that even though A was obtained empirically, it is not a coincidence that $A \approx e^{\gamma} \frac{6}{\pi^2} = 1.08276 \dots$.

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