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**MR2113535 (2006b:11111)**
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**Discrete limit theorems for general Dirichlet series. III. (English summary)**
*Cent. Eur. J. Math.* **2** (2004), *no. 3,* 339–361 (*electronic*).
11M41 (30B50 60B10)



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For two sequences $\{a_n : n \in \mathbb{N}\}$ and $\{\lambda_n : n \in \mathbb{N}\}$ of complex numbers, where the second sequence is monotone and converging to $+\infty$, one defines a general Dirichlet series as

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s},$$

where $s = \sigma + it$ is a complex variable. Let $f(s)$ denote the sum of this series. Then for some $\sigma_a$ the series will be absolutely convergent in the region $\sigma > \sigma_a$. Now, assume that everywhere in the region of its meromorphic continuation (i.e. in $\sigma > \sigma_1$, where $\sigma_1 < \sigma_a$) the poles of $f(s)$ belong to a compact set and $f(s)$ itself satisfies the following conditions:

(1) $f(\sigma + it) = O(|t|^\alpha)$, for all $|t| \geq t_0, \alpha > 0$;
(2) $\frac{1}{2T}\int_{-T}^{T} |f(\sigma + it)|^2 dt = O(T)$, as $T \to \infty$;
(3) $\lambda_n \geq c(\log n)^\delta$, for some positive constants $c, \delta$.

In 2003, in the first paper in this sequence [Part I, Chebyshevskiĭ Sb. **4** (2003), no. 3(7), 156–170; MR2051602 (2005a:11138)], the authors showed that if conditions (1) and (2) are satisfied, and $\{\lambda_n\}$ is a sequence of linearly independent algebraic numbers over $\mathbb{Q}$, then for any fixed $h > 0$, the probability measure $Q_N(A)$ defined as

$$Q_N(A) := \mu_N(f(s + imh) \in A), \quad A \in \mathfrak{B}(\mathbb{C}),$$

converges weakly to $Q_f$ as $N \to \infty$. Here $\mathfrak{B}(\mathbb{C})$ denotes the class of Borel sets of $\mathbb{C}$.

In the second paper in the sequence [Part II, Liet. Mat. Rink. **44** (2004), no. 1, 85–92; MR2116493 (2005h:11202)], the second author showed that this result could be generalized as follows. For $\sigma > \sigma_1$, consider the probability space $(\Omega, \mathfrak{B}(\Omega), m_H)$, where $\Omega$ is the infinitedimensional torus (a compact topological abelian group) equipped with the product topology, and $m_H$ is the Haar measure on $(\Omega, \mathfrak{B}(\Omega))$. On this probability space one defines a complex random element $f(\sigma,\omega)$ by

$$f(\sigma,\omega) := \sum_{n=1}^{\infty} a_n \omega(m) e^{-\lambda_n \sigma},$$

and denotes its distribution by $Q_f$. Then, just like above, the probability measure

$$\widehat{Q}_N(A) := \mu_N(f(s+imh) \in A), \quad A \in \mathfrak{B}(H(D_a)),$$

converges weakly to $\widehat{Q}_f$ as $N \to \infty$.

The paper under review is the third paper in this sequence, and it continues with the above theme, employing related methods and techniques. The main theorem can be summarized as follows. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and $d$ a certain metric compatible with its topology. Consider $M(G)$, the space of meromorphic functions $g\colon G \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. If all three conditions (1), (2) and (3) are satisfied, and $\{\lambda_n\}$ is a sequence of linearly independent algebraic numbers over $\mathbb{Q}$, then once again

$$P_N(A) := \mu_N(f(s+imh) \in A), \quad A \in \mathfrak{B}(M(D)),$$

converges weakly to $P_f$ as $N \to \infty$.

**Reviewed** by *Filip Saidak*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.


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