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An asymptotic formula for a sum involving zeros of the Riemann zeta-function. (English summary)

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Let us define the well-known Chebyshev and von Mangoldt functions via

$$\psi(x) := \sum_{n\le x} \Lambda(n) = \sum_{p^m \le x} \log p.$$

The prime number theorem states that $\psi(x) \sim x$ as $x \to \infty$. It turns out that the error we commit in this approximation can be given explicitly in terms of zeros of the Riemann zeta function. In 1895, von Mangoldt proved the following version of Riemann's explicit formula:

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} + \frac{1}{2}\log\left(\frac{x^2}{x^2-1}\right) - \log 2\pi,$$

where the sum is extended over all nontrivial zeros of $\zeta(s)$, counted with multiplicity. In 1911, Landau showed that, for all $x > 1$, we also have

$$\sum_{0<|\mathrm{Im}\,\rho|\le T} x^\rho = -\frac{T}{2\pi}\Lambda(x) + O(\log T).$$

In the paper under review the authors prove a smoothed version of Landau's formula. One of several interesting special cases is the following: As $u \to +0$,

$$\sum_\rho e^{u\rho^2 - u\rho} = F(u) + O(1),$$

where the sum is extended over all nontrivial zeros of $\zeta(s)$, counted with multiplicity, and the function $F(u)$ is defined as

$$F(u) := \frac{1}{\sqrt{16\pi u}} \log \frac{1}{u} - \frac{\log(16\pi^2) + \gamma}{\sqrt{16\pi u}},$$

with γ denoting Euler's constant. Assuming the Riemann Hypothesis, this implies

$$\sum_{0<|\operatorname{Im}\rho|\le T} e^{-u(\frac{1}{4}+(\operatorname{Im}\rho)^2)} = F(u) + O(1),$$

or equivalently

$$-2\int_0^\infty N(T)d\left(e^{-uT^2}\right) = F(u) + O(1),$$

where

$$N(T) := \sum_{0<|\operatorname{Im}\rho|\le T} 1.$$

The main results of the paper are proved by a suitable choice of "test" function in Weil's explicit formula, which directly connects primes and zeros of $\zeta(s)$ in the critical strip.

Reviewed by *Filip Saidak*