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Li, Shuguang [Li, Shuguang[1]] (1-HIH-NS)
An improvement of Artin's conjecture on average for composite moduli.
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In 1927 E. Artin stated a famous conjecture saying that if we let $P_a(x)$ be the number of primes $\leq x$, for which a is a primitive root, then the limit $\lim_{x\to\infty} P_a(x)/\pi(x)$ always exists. Forty years later, in 1967, C. Hooley [J. Reine Angew. Math. **225** (1967), 209–220; MR0207630 (34 #7445)] showed that a certain GRH implies the truth of this intriguing statement. Unconditionally, the problem is still open another four decades later, but already in 1969, P. J. Stephens [Mathematika **16** (1969), 178–188; MR0498449 (58 #16565)] showed that Artin's primitive root conjecture is true on average, i.e. that for all $y > \exp(4\sqrt{\log x \log\log x})$, we have

$$\frac{1}{y}\sum_{a\leq y} P_a(x) = \alpha \mathrm{Li}(x) + O\left(\frac{x}{(\log x)^B}\right),$$

where $B > 1$ and α is the Artin constant:

$$\alpha = \prod_p \left(1 - \frac{1}{p(p-1)}\right) = 0.37395\ldots.$$

Naive attempts to extend all these results and conjectures to composite moduli are easily shown to lead to incorrect conclusions, and the theory concerning $N_a(x)$—the number of all positive integers $\leq x$ for which a is a primitive root—happens to be quite a bit more intricate. (A recent work of the author and C. Pomerance [J. Reine Angew. Math. **556** (2003), 205–224; MR1971146 (2004c:11177)] contains several important contributions to this subject.) In the present paper the author employs Stephens' old method (which, in turn, was motivated by Turán's work) in order to prove the following theorem: Let $R(n)$ denote the number of primitive roots modulo n in the interval $[1, n]$, and let $x, y \geq 3$. Then

$$\frac{1}{y}\sum_{a\leq y} N_a(x) = \sum_{n\leq x}\frac{R(n)}{n} + O\left(\frac{x^{3/2}}{y}\exp\left(2.12\sqrt{\frac{\log x}{\log\log x}}\right)\right).$$

It is also shown that in situations where $y \geq \exp\left((\log x)^{3/4}\right)$, one can further improve the error term and get a more versatile estimate

$$\frac{1}{y}\sum_{a\leq y} N_a(x) = \sum_{n\leq x}\frac{R(n)}{n} + O\left(x\exp\left(-\frac{5}{16}\sqrt{\log x}\right)\right).$$

Reviewed by *Filip Saidak*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.