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New Erdős-Kac type theorems. (English summary)

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The objective of this paper is a new proof, under a quasi-Generalized Riemann Hypothesis (quasi-GRH), of the prime analogue of an Erdős-Kac-type theorem which was formulated as a conjecture by P. Erdős and C. Pomerance [Rocky Mountain J. Math. **15** (1985), no. 2, 343–352; MR0823246 (87e:11112)]. The original conjecture was recently proved (also under a quasi-GRH) by Saidak ["Non-abelian generalizations of the Erdős-Kac theorem", Ph.D. thesis, Queen's Univ., Kingston, ON, 2001, available at www.collectionscanada.ca/obj/s4/f2/dsk3/ftp05/NQ63451.pdf], and by M. R. Murty and Saidak [Canad. J. Math. **56** (2004), no. 2, 356–372; MR2040920 (2005a:11114)], who also established (under a quasi-GRH) its prime analogue. The result proved can be stated as follows. Let $a > 1$ be an integer, and for p prime let $f_a(p)$ be the order of a modulo p. Then for any $\alpha < \beta$ we have

$$\left| \left\{ p \le x : \alpha \le \frac{\omega(f_a(p)) - \log\log p}{(\log\log p)^{1/2}} \le \beta \right\} \right| \sim \Phi(\alpha, \beta)\pi(x)$$

as $x \to \infty$, where $\Phi(\alpha, \beta) := (2\pi)^{-1/2} \int_\alpha^\beta e^{-t^2/2} dt$.

The proof given by Murty and Saidak [op. cit.] appeals to an Erdős-Kac type theorem on $\omega(p-1)$ due to Halberstam, whereas the present proof appeals to the moments of $\omega(f_a(p))$ and to the powerful Fréchet-Shohat theorem [M. R. Fréchet and J. A. Shohat, Trans. Amer. Math. Soc. **33** (1931), no. 2, 533–543; MR1501604; Zbl 0002.28003].

Reviewed by *Y.-F. S. Pétermann*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.