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Non-abelian generalizations of the Erdős-Kac theorem. (English summary)
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The main object of this paper is the proof, under an assumption slightly weaker than the generalized Riemann hypothesis (GRH), of an Erdős-Kac type theorem which was formulated as a conjecture by P. Erdős and C. Pomerance [Rocky Mountain J. Math. **15** (1985), no. 2, 343–352; MR0823246 (87e:11112)], and which can be stated as follows. Let $a>1$ be an integer, and for $(n,a)=1$ let $f_a(n)$ be the order of a modulo n. Then for any $\alpha<\beta$ we have

$$\left|\left\{n\le x\colon (a,n)=1, \alpha\le \frac{\omega(f_a(n))-\frac{1}{2}(\log\log n)^2}{\frac{1}{\sqrt{3}}(\log\log n)^{3/2}}\le\beta\right\}\right| \sim \Phi(\alpha,\beta)\frac{\varphi(a)}{a}x$$

as $x\to\infty$, where $\Phi(\alpha,\beta):=(2\pi)^{-1/2}\int_\alpha^\beta e^{-t^2/2}dt$, and where φ denotes the Euler function.

The authors also derive, under the same "quasi-GRH", an Erdős-Kac type theorem for $\omega(f_a(p))$ (p prime, $p\le x$), as well as—though less precise or more complicated to state, but without the assumption of any unproven hypothesis—Erdős-Kac type theorems for $\omega(f_a(n))$ ($n\le x$).

The adjective "nonabelian" of the title refers to the fact that the extensions $L_q:=\mathbb{Q}(\omega_q,\sqrt[q]{a})$ intervening in the study of $f_a(n)$, where q is a prime and ω_q denotes a primitive q-th root of unity, are not abelian extensions. And, more precisely, the "quasi-GRH" assumed by the authors is the assumption that the Dedekind zeta functions $\zeta_q(s)$ of L_q have no zero in a region $\mathrm{Re}(s)>\theta$ for some $\theta<1$.

Reviewed by *Y.-F. S. Pétermann*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.