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**Banks, W. D.** (1-MO); **Luca, F.**; **Saidak, F.** (1-WKFR); **Stănică, P.**
**Compositions with the Euler and Carmichael functions. (English summary)**
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Let $\varphi$ and $\lambda$ stand respectively for the Euler function and the Carmichael function. Recall that the latter one is defined as the maximal order of any element in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$. More explicitly, given a prime power $p^\alpha$ with $p \neq 2$, then $\lambda(p^\alpha) = p^{\alpha-1}(p-1)$, while $\lambda(2) = 1$, $\lambda(4) = 2$ and $\lambda(2^\alpha) = 2^{\alpha-2}$ if $\alpha \geq 3$, and for an arbitrary integer $n \geq 2$ with prime factorization $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then $\lambda(n) = \mathrm{LCM}[\lambda(p_1^{\alpha_1}), \ldots, \lambda(p_k^{\alpha_k})]$; also, $\lambda(1) = 1$. Observe that in contrast with the Euler function, the Carmichael function is not multiplicative. The authors investigate the composite functions $\varphi \circ \lambda$ and $\lambda \circ \varphi$ by establishing lower and upper bounds for the counting function of the set

$$\mathcal{A}(x) := \{n \leq x \colon \varphi(\lambda(n)) = \lambda(\varphi(n))\}.$$

More precisely, they prove that there exist positive constants $C$ and $x_0$ such that, for all $x \geq x_0$,

$$\#\mathcal{A}(x) \geq \exp\{C \log x / \log\log\log x\},$$

and also that the inequality

$$\#\mathcal{A}(x) \leq x/(\log x)^{3/2+o(1)}$$

holds as $x \to \infty$. This leads them to conjecture that

$$\#\mathcal{A}(x) = x/(\log x)^{2+o(1)}$$

as $x \to \infty$. Moreover, they show that the estimate

$$\varphi(\lambda(n))/\lambda(\varphi(n)) = \exp\{(1+o(1))(\log\log n)^2 \log\log\log n\}$$

holds on a set of positive integers $n$ of asymptotic density one.

Reviewed by *Jean-Marie De Koninck*

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*Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.*