MR1906418 (2003h:11119) 11N56 (11N37)
Saidak, Filip (3-CALG)
The normal number of prime factors of $f_a(n)$. (English summary)
J. Ramanujan Math. Soc. **17** (2002), *no. 1,* 19–33.

Denote as usual by $\omega(n)$ the number of prime factors of the positive integer n counted without multiplicity. The function $\omega(n)$ has "normal" order $\log\log n$ and a famous theorem of Turán bounds the variance of this distribution.

For an integer a relatively prime to n define its order $f_a(n)$ to be the smallest positive integer m such that $a^m \equiv 1 \pmod{n}$. The author is interested in obtaining results of Turán type for the function $\omega(f_a(n))$.

The author assumes a quasi-Riemann hypothesis for the Dedekind zeta functions of certain nonabelian number fields. Subject to this hypothesis he proves, for each squarefree integer $a \geq 2$,

Theorem 1:

$$\sum_{\substack{p\leq x\\(p,a)=1}} \Big(\omega(f_a(p)) - \log\log p\Big)^2 \ll \pi(x)\log\log x$$

and Theorem 2:

$$\sum_{\substack{n\leq x\\(n,a)=1}} \Big(\omega(f_a(n)) - \frac{1}{2}(\log\log n)^2\Big)^2 \ll x(\log\log x)^3.$$

Very roughly speaking, one uses the conjecture to prove that primes in a certain Kummer extension attached to the integer a split with frequency as expected and as a result on average $\omega(f_a(p))$ is not so different from $\omega(p-1)$. On the other hand, if on the left-hand side of Theorem 1 one replaces $\omega(f_a(p))$ by $\omega(p-1)$, then on expanding the square and interchanging the order of summation we are led to sums which can be treated by the Bombieri-Vinogradov theorem. As a result we get Theorem 1. The proof of Theorem 2 is rather more complicated and uses amongst other things several lemmata concerning the distribution over shifted primes $n = p - 1$ of the truncated function $\Omega_y(n) = \sum_{p^\alpha\|n, p^\alpha<y} \alpha$, due to P. Erdős and C. Pomerance [Rocky Mountain J. Math. **15** (1985), no. 2, 343–352; MR0823246 (87e:11112)].

The author mentions that slightly different versions of the proofs appear in his Ph.D. thesis ["Non-Abelian generalizations of the Erdős-Kac theorem", Queen's Univ., Kingston, ON, 2001].

Reviewed by *John B. Friedlander*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.