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On the modulus of the Riemann zeta function in the critical strip. (English summary)

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It is well known [E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second edition, Oxford Univ. Press, New York, 1986; [MR0882550 \(88c:11049\)](#)] that the function

$$\chi(s) = 2^s \pi^{s-1} \sin \frac{s\pi}{2} \Gamma(1-s)$$

appearing in the functional equation of the Riemann zeta-function, $\zeta(s) = \chi(s)\zeta(1-s)$, satisfies (using the usual notation $s = \sigma + it$)

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-\frac{1}{2}} e^{i(t+\frac{\pi}{4})} \left(1 + O\left(\frac{1}{t}\right)\right).$$

In this paper the authors prove for $\frac{1}{2} \leq \sigma \leq 1$ and $t \geq 2\pi + 1$ the inequality $|\zeta(1-s)| \geq |\zeta(s)|$, and sharpen the estimate for $\chi(s)$ by showing that

$$\left(\frac{|s|}{2\pi}\right)^{\sigma-\frac{1}{2}} \left(1 - \frac{4\sigma^3 - \sigma}{12t^2}\right) \leq \frac{1}{|\chi(s)|} \leq \left(\frac{t}{2\pi}\right)^{\sigma-\frac{1}{2}} \leq \left(\frac{|s|}{2\pi}\right)^{\sigma-\frac{1}{2}}.$$

Their argument depends on elementary considerations involving uses of relevant series expansions to a sufficient number of terms. Open problems and conjectures (some of which imply the Riemann Hypothesis), and some numerical data are also supplied in the paper.

Reviewed by *Cem Y. Yildirim*

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