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Values of arithmetical functions equal to a sum of two squares. (English)

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Y. Motohashi [Acta Math. Acad. Sci. Hung. 22, 207–210 (1970; Zbl.0225.10044)] showed that $N(x) \ll x\log^{-3/2} x$, where $N(x)$ denotes the number of primes not exceeding x which are of the form $a^2 + b^2 + 1$ $(a, b \in \mathbb{Z})$ and conjectured that in fact $N(x)$ is asymptotic to $Cx\log^{-3/2} x$ as $x \to \infty$. The present authors show that

$$M(x) \asymp x\log^{-3/2} x, \tag{1}$$

where $M(x)$ denotes the number of $n \le x$ such that $\varphi(n) = a^2 + b^2$ $(a, b \in \mathbb{Z})$, $\varphi(n)$ is the Euler function, and $A \asymp B$ means that $C_1A \le B \le C_2B$ for suitable constants $0 < C_1 < C_2$. A result analogous to (1) holds if $\varphi(n)$ is replaced by

$$\psi(n) = n\prod_{p|n}(1 + 1/p), \quad \sigma(n) = \sum_{d|n} d,$$

i.e., the Dedekind function and the sum of divisors function. In all three cases it is plausible to conjecture, similarly to $N(x)$, that the counting function in question is asymptotic to $Cx\log^{-3/2} x$ as $x \to \infty$ with a suitable $C > 0$ in each case. Proving these conjectures, however, seems to be quite hard. The proofs of the authors' results are based on five lemmas, the first of which improves an estimate of *H. Iwaniec* [Acta Arith. 21, 203-224 (1974; Zbl.0271.10043)], namely

$$\#\{p \le x : p - 1 \in mS\} \ll \frac{x}{\varphi(m)(\log x)^{3/2}} \quad (1 \le m \ll (\log x)^{3/2}), \tag{2}$$

where m is a squarefree number whose prime factors are of the form $4k + 3$, while S is the set of integers representable as a sum of two integer squares. The authors' Lemma 1 improves (2) by showing that the quantity on the left-hand side is, for all m,

$$\ll \frac{x}{\varphi(m)(\log(x/m))^{3/2}}.$$

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