Let $p$ be a prime. Every positive integer can be written as a finite base-$p$ expansion. For example the base 5 expansion of 1776 is $2 \cdot 5^4 + 4 \cdot 5^3 + 1 \cdot 5^2 + 0 \cdot 5 + 1 \cdot 5^0$

A $p$-adic number is an infinite base $p$ expansion that may include finitely many negative powers of $p$. The collection of all $p$-adic numbers is denoted $\mathbb{Q}_p$. The $p$-adic valuation $\nu_p(n)$ is the lowest exponent of $p$ in the base $p$ expansion of $n$. For example $\nu_p(1776) = 0$. On the other hand, $\nu_p(400) = 2$ because $400 = 1 \cdot 5^2 + 3 \cdot 5^1$.

A monic polynomial of degree $n$ has the form

$$p(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_1x + c_0$$

A root of $p(x)$ is a number $r$ such that $p(r) = 0$. For example, the quadratic $x^2 + bx + c$ in terms of $b$ and $c$:

$$r = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

The discriminant of a polynomial is the product of all the differences of its roots. For instance the discriminant of the quadratic $x^2 + bx + c$ is $b^2 - 4c$.

For a polynomial $p$ of degree $n$ with coefficients in $\mathbb{Q}_p$, its $j$-invariant is defined by

$$\nu_p(\text{disc}(p)) = n + j - 1$$

There are only finitely many distinct polynomials with $p$-adic coefficients of a given degree. This project studied an invariant of degree $p^2$ polynomials over $\mathbb{Q}_p$, for an odd prime $p$.

**Ramification Polygons**

The Newton polygon of $p(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1x + c_0$ is the lower convex hull of the set of points $(i, \nu_p(c_i))$.

Let $r$ be a root of $p$. The ramification polygon of $p$ is the Newton polygon of the ramification polynomial $p(x) = r^{-n}p(x + r)$.

For example, consider the polynomial $p(x) = x^3 + 2$ over the field $\mathbb{Q}_2$. The associated ramification polynomial is $\rho(x) = x^3 + 8x^2 + 28x + 56x^3 + 70x^4 + 56x^5 + 28x^2 + 8x + 1$.

**Research and Methods**

In the case of degree $p^2$ extensions of $\mathbb{Q}_p$, there are the two possible shapes that a ramification polygon can have.

We defined a function $n(j)$ that returns the number of distinct ramification polygons for a given $j$-value. In addition we found the total number of different ramification polygons over all possible $j$-values.

**The Function $n(j)$**

Let $j = a_3p^3 + a_2p + a_0$ be the base $p$ expansion of $j$. Then $n(j)$ is defined by

$$n(j) = \begin{cases} a_3 + 6 & \text{if } j < p^2 \\ a_2 + 1 & \text{if } p^2 < j < p(p + 1) \\ 1 & \text{if } j \geq p(p + 1) \text{ and } p \mid j \\ a_1 + 1 & \text{if } j > p(p + 1) \text{ and } p \mid j \end{cases}$$

where $\delta = 0$ if $a_0 < a_1$ and $\delta = 1$ otherwise.

**Visualization of $n(j)$**

In the graphs below, “# Ram. Pgons” refers to the number of distinct ramification polygons.

**Classification And Implications**

Let $R_j$ denote the complete set of all distinct ramification polygons for a given $j$-value. As before let $j = a_3p^3 + a_2p + a_0$ be the base $p$ expansion of $j$ and set $\delta = 0$ if $a_0 < a_1$ and $\delta = 1$ otherwise. The table below provides a classification of all possible ramification polygons for possible $j$-values.

$$\begin{array}{c|c|c} j & R_j \\ \hline 1 \leq j < p^2 & \{((1, j), (p^2, 0))\} & 1 \leq c < a_1 + \delta \\ p^2 \leq j < p(p+1) & \{((1, j), (p^2, 0))\} & 1 \leq c < p \\ p(p+1) \leq j \leq 2p^2 \text{ and } p \mid j & \{((1, j), (p^2, 0))\} & a_0 < c \leq p \\ \end{array}$$

Observe that $n(j) = |R_j|$. Consequently the following hold:

1. $|R_j| \in \{1, 2, \ldots, p\}$.

2. The number of distinct ramification polygons for a given $j$-value is summarized below:

   - $|R_j| = 1$ for $3p - 2$ different $j$-values.
   - $|R_j| = p$ for $p$ different $j$-values.
   - $|R_j| = k$ for $2p - 1$ different $j$-values.

3. The total number of distinct ramification polygons is $p^3 - \frac{p^2 - 3p}{2} - 1$.

**Impact And Future Research**

While there are only finitely many polynomials of given degree with $p$-adic coefficients, a complete classification of these polynomials and their invariants has not been completed in generality. Polynomials of degree $p$ and $2p$ were classified by Amano [1] and Awtrey-Hadgis [2], respectively. For all other degrees divisible by $p$, the only known cases are when the degree $\leq 15$. So nothing is known for degree $p^2$ beyond $p = 3$.

We observed that the number of non-isomorphic generating polynomials for a given $j$-value, could be expressed as

$$(p - 1) + (n(j) - 1)(p - 1)^2.$$

**References**


Eisenstein equations of degree $p$ in a $p$-adic field.


Totally ramified $p$-adic fields of degree $2p$. submitted.

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